Supermaps on generalised theories

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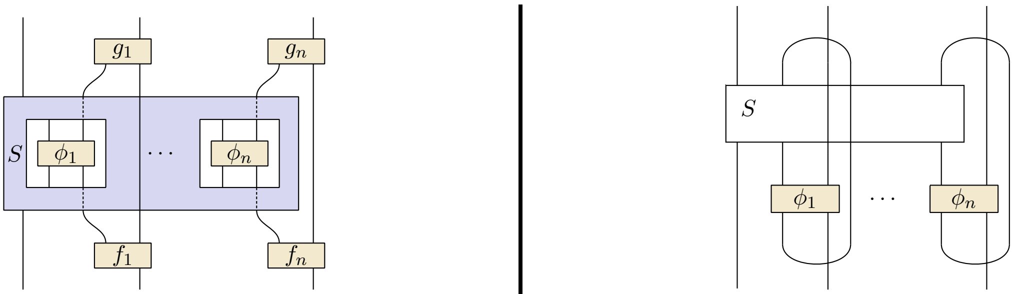
1 Generalising higher-order quantum operations

Fundamental to the study of causal and indefinite-causal structure in quantum information theory is the notion of a higher-order quantum operation [3, 10, 12]. In light of the utility of the higher-order framework, there have been a variety of attempts to generalise higher-order processes beyond finite-dimensional classical and quantum theory to broader classes of probabilistic theories, and to infinite-dimensional quantum theories, ranging from general categorical [6, 7, 13, 14] and information-theoretic [2, 9] approaches to concrete, specialised ones [1, 4, 5, 11].

Categorical approaches to the definition of higher-order maps in quantum theory have typically fallen into two camps: one that relies quite directly on an abstraction of the (finite-dimensional) channel-state duality of the CJ isomorphism (compact closure) [7, 13, 14], and one that attempts to define supermaps across a broader class of theories [6, 13, 14] (including those which are infinite-dimensional) by referring only to circuit-theoretic composition rules (basic compatibility with notions of space and time).



(a) A general categorical supermap [6, 13, 14] is encoded as a family of functions (referred to as a locally applicable transformation) that can be applied to parts of bipartite processes - an abstract interpretation of the completeness of completely positive maps.

(b) CJ-supermaps are a special class of categorical supermaps which can be implemented concretely using a channel and a contraction along cups and caps of a broader compact closed category into which the theory embeds [7, 13, 14].

Figure 1: Two approaches to defining supermaps on generalised theories (such as operational and categorical probabilistic theories). In each case a multi-input categorical supermap of type $S : \bigtimes_i (A_i \Rightarrow A_i') \to (B \Rightarrow B')$ defines a family of functions $S_{X_1 X_1' X_2 X_2'} : \bigtimes_i \mathcal{C}^{\mathbf{d}}(A_i \otimes X_i, A_i' \otimes X_i') \to \mathcal{C}^{\mathbf{d}}(B \otimes_i (X_i), B' \otimes_i (X_i'))$.

The general categorical approach has aimed to give a stable unifying framework for the study of causal structure beyond finite-dimensional quantum theory, and so, it is natural to consider to what extent the general categorical approach, the restricted finite-dimensional categorical approach, and more specialised approaches [1, 11] in the literature coincide. In other words, we ask whether the study of causal structure beyond finite-dimensional quantum theory is stable.

A particularly useful test of the categorical approach is the recent construction of a higher-order theory for Boxworld [1] in which physical subtleties regarding the nature of signalling through higher-order processes were identified: the authors argue that a physically meaningful notion of supermap on Boxworld operations, in particular the one corresponding to a process matrix, requires a constraint termed 'No-Signalling Without System Exchange (NSWSE)'. In order to test the applicability of the categorical approach to generalising higher-order transformations, we consider not only whether they coincide on a broad class of theories, but also whether, within that class, their application to Boxworld incorporates the NSWSE requirement.

2 The Yoneda lemma for supermaps on generalised theories

The passage from abstract families of functions satisfying commutativity properties to concrete representations is one way to interpret the fundamental result of category theory: the Yoneda lemma [8]. In technical language, this lemma states that natural transformations between representable presheaves on a category correspond to morphisms in the category. Concretely, it means that when a family of functions on processes commutes with pre-composition, it can be represented by post-composition with a process. While there cannot be a Yoneda Lemma for categorical supermaps on arbitrary monoidal categories due to the results of [13], we consider whether there exists some restricted physically-motivated class of theories on which a Yoneda-inspired lemma might hold.

To this end, we consider a generalised theory to be a symmetric monoidal category equipped with a set of wires which represent the classical interface. This allows for the extraction of probabilistic correlations from compositional experimental set-ups. As a basic outline of how such theories can be used, we show a few diagrams that can be written within such theories and give their operational interpretation. Examples of generalised theories include: classical theory, quantum theory, boxworld, real quantum theory, and quantum theory over a broader class of semirings.

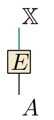
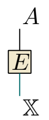
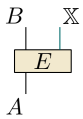
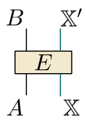
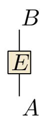
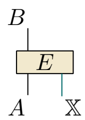
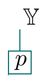
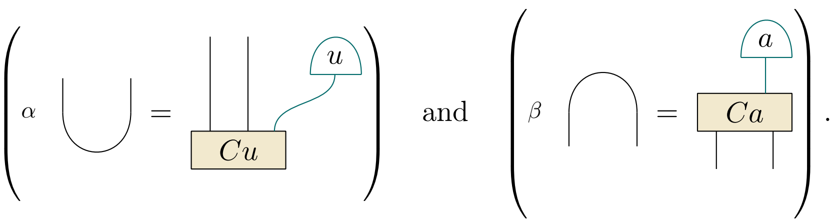
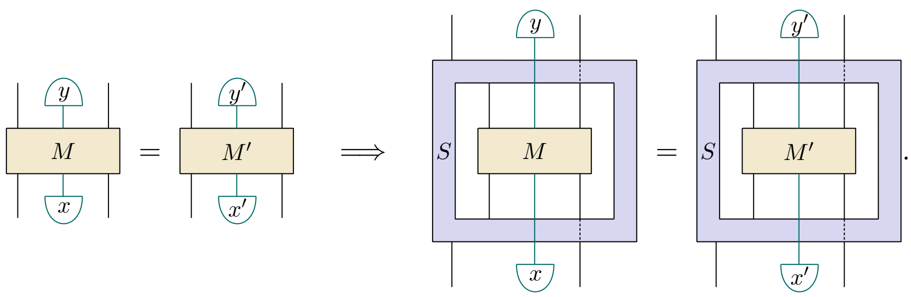
Destructive Measurement		Controlled State Preparation	
Non-destructive Measurement		Classical Process	
Conditional Measurement		Physical Process	
Controlled Process		Probability Distribution	

Figure 2: Generalised theories are theories of physical processes equipped with classical interfaces. Also included within the definition of generalised theories is the existence of a notion of classical control for processes.

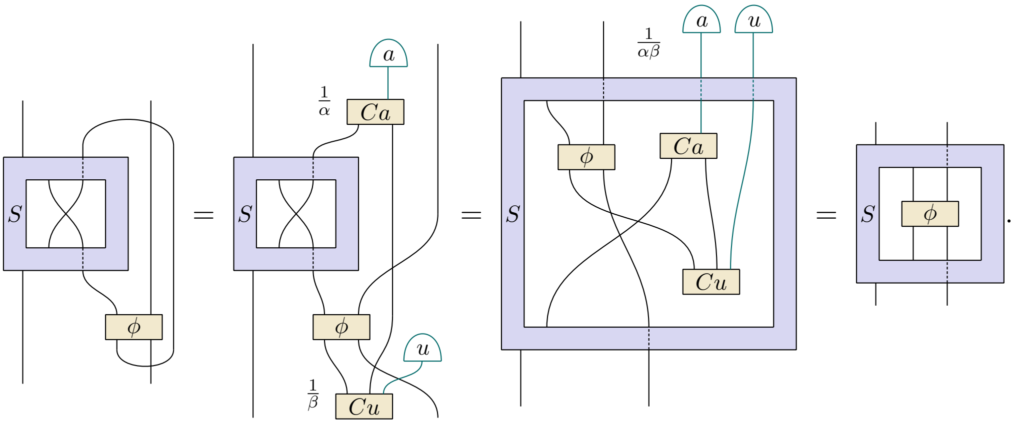
Since generalised theories are symmetric monoidal categories, higher-order processes on them can be defined immediately as the locally applicable transformations (categorical supermaps). However, the example theories we have presented have another property in common that allows CJ-supermaps to be defined. A generalised theory has channel-state duality if its non-deterministic theory $\mathcal{C}$ is a compact closed category in which the cup $\cup$ and cap $\cap$ are (up to scalars) instrument elements:



In order to prove a Yoneda-inspired lemma for supermaps, we first leverage commutativity over classical-interface wires to confirm that categorical supermaps are non-contextual assignments on instrument components:

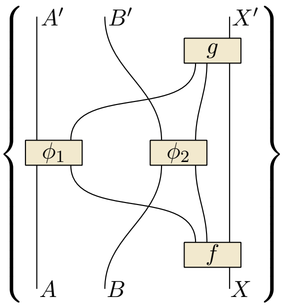


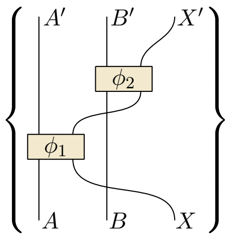
The structure of the proof of the Yoneda lemma in terms of these components is then straightforward:



The multi-input case follows similarly, using the fact that we have commutativity separately in each input to S. Consequently, on essentially any theory for which it makes sense to compare locally applicable transformations and CJ-supermaps, they coincide.

Extension to general higher-order objects: The results of this paper can be generalised to supermaps on general types of processes encoded as concrete profunctors. A concrete profunctor $\mathcal{P}$ is a pair of objects (A, A') along with for each pair of objects X, X' a chosen subset $\mathcal{P}(X, X') \subseteq \mathcal{C}(A \otimes X, A' \otimes X')$, closed under action by combs on the environment, and including the swap morphism in $\mathcal{P}(A', A)$. Examples of concrete profunctors include, spaces of product channels with shared past and future, and spaces of channels which allow for one-way communication

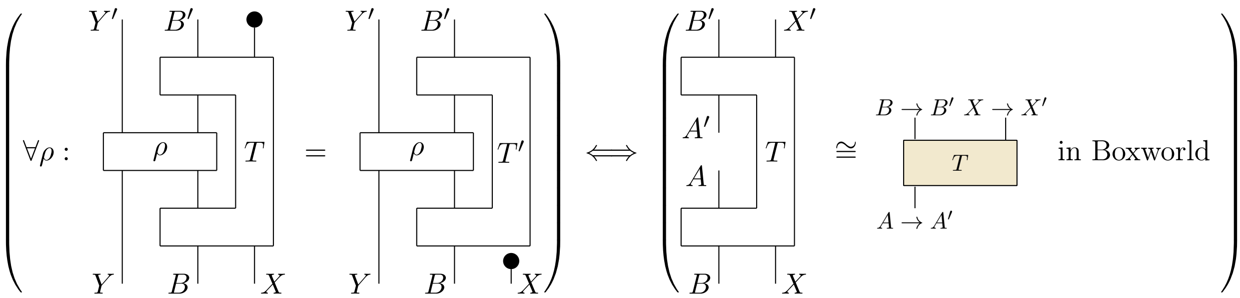
$\mathcal{P}(X, X') =$ 

$\mathcal{P}(X, X'), =$  .

For any pair of concrete profunctors on a generalised theory, the categorical supermaps of type $\mathcal{P} \to \mathcal{Q}$ are equal to the CJ-supermaps of the same type.

Higher-order real quantum theory: As a direct application, we note that real quantum theory is a generalised theory with channel-state duality. This allows us to construct a stable model of higher-order real quantum theory.

NSWSE is a principle of categorical supermaps: The most direct proposal for defining supermaps in general physical theories is the Boxworld construction of [1], which imposes the principle of *no signalling without system exchange* (NSWSE). The definition of NSWSE tensors hinges on a notion of no-signalling for Boxworld instruments. By observing that this notion is equivalent to expressibility in terms of the monoidal product in Boxworld:

 ,

we show that the categorical-supermaps approach encodes the NSWSE requirement as a purely compositional principle; more precisely, categorical supermaps on Boxworld coincide with the NSWSE Boxworld tensors.

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