A categorical approach to control Lyapunov and control barrier functions *

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Stability and safety are central concerns in control theory. A control Lyapunov function (CLF) certifies that a controller exists which drives a system to a desired equilibrium. Artstein [5] proved that the existence of a CLF implies the existence of a stabilizing controller, and Sontag [11] made this constructive for control-affine systems, giving an explicit formula. Control barrier functions (CBFs) [4] play an analogous role for safety: a CBF certifies that a safe set is forward invariant under some admissible controller, with the underlying closed-loop guarantee given by Nagumo's theorem [1]. Together, CLFs and CBFs form the foundation of safety-critical control. We build on the coalgebraic Lyapunov framework of [3, 2] to develop a general categorical account of safe and stabilizing controller synthesis, recovering classical results as special cases and obtaining new ones in settings not previously accessible to these methods.

We work in the setting of control coalgebras:

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & \mathcal{F}(B) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_B} & \\ & B & \end{array}$$

a bundle $p\colon E \to B$ over a state space B, together with a map $f\colon E \to \mathcal{F}B$ that is a bundle morphism into a canonical bundle over $\mathcal{F}B$. The fiber over a state $x \in B$ parametrizes the admissible control inputs, and a controller is a section of p. Here we work in an arbitrary category with finite products and an endofunctor $\mathcal{F}$ with a canonical projection $\pi\colon \mathcal{F} \Rightarrow id$. This structure appears in Tabuada–Pappas [12] in the setting of manifolds. In the case when $E = B$ and $p = \mathrm{id}_B$, then a control coalgebra is an ordinary "autonomous" coalgebra [10] which is a section of the canonical bundle. Control coalgebras are also closely related to lenses [8].

In this setting we prove an analog of Nagumo's theorem, ensuring that "barrier morphisms" are safety certificates for closed-loop systems. We then define the analogs of control Lyapunov functions and control barrier functions, and prove that their existence implies the existence of a stabilizing or safe controllers, respectively.

This framework recovers classical controller synthesis results, including those for CBFs on Euclidean spaces, geometric CBFs on manifolds [7], and matrix CBFs [9] as special cases by varying the ambient category and measurement object. Further instantiations recover contraction theory [6] and yield new CLF and CBF results for G-equivariant control systems.

*Joint work with Aaron Ames.

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