Hybrid Systems as coalgebras[*]

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Hybrid dynamical systems combine continuous flows with discrete transitions: a system evolves according to a vector field until it hits a guard set, at which point a reset map instantaneously updates the state, and the process repeats. They arise throughout science and engineering, and their stability theory has developed piecemeal, producing separate Lyapunov-like results for different notions of stability. We provide a unified account using the language of $\mathcal{F}$-coalgebras [6].

The central observation is that a Lyapunov function is a morphism from a system into a simple stable target system σ, and different notions of stability correspond to different choices of σ. To make this precise, we express hybrid systems as coalgebras of an endofunctor $\mathcal{H}$ on a category Chart that blends smooth manifolds and sets, building on the framework of categorical systems theory [4]. Solutions to a hybrid system correspond to coalgebra morphisms out of a canonical clock coalgebra on hybrid time domains. The Lyapunov conditions fall out of a single lax commuting diagram, and a collection of known stability results in hybrid systems theory follow as corollaries by choosing appropriate target systems σ. We apply this to Lagrangian hybrid systems, a broad class of mechanical systems with impacts.

$$\frac{\partial V}{\partial x}\cdot f(x) \le \sigma(x) \implies \begin{array}{ccc} E & \xrightarrow{V} & R \\ {\scriptstyle f}\big\downarrow & \swarrow\!\!\swarrow & \big\downarrow{\scriptstyle \sigma} \\ \mathcal{F}(E) & \xrightarrow[\mathcal{F}V]{} & \mathcal{F}(R) \end{array}$$

$$\implies \begin{array}{ccc} \binom{S}{M} & \overset{V}{\rightrightarrows} & \binom{\mathbb{R}}{\mathbb{R}} \\ {\scriptstyle f}\downdownarrows & \swarrow\!\!\swarrow & \downdownarrows{\scriptstyle \sigma} \\ \mathcal{H}\binom{S}{M} & \underset{\mathcal{H}V}{\rightrightarrows} & \mathcal{H}\binom{\mathbb{R}}{\mathbb{R}} \end{array} \implies \{(V_d(s^+, x^+), V_c(x^+))\} \le \sigma_d(V_d(s,x), V_c(x))$$

Prior work [5] has expressed hybrid automata coalgebraically, but the coalgebras used there encode collections of hybrid trajectories rather than the dynamical systems themselves. This distinction is critical for Lyapunov theory, which requires access to the structure of the system, not merely its behavior. Our $\mathcal{H}$-coalgebras encode the dynamics directly: the continuous vector fields, the guard sets, and the reset maps are all present in the coalgebra structure, and solutions arise as coalgebra morphisms out of a clock coalgebra rather than being built into the states.

One stability notion requires special treatment: Zeno stability, where infinitely many discrete transitions accumulate in finite time. Unlike the other stability notions, Zeno stability cannot be captured by a single Lyapunov morphism into a one-component target system. It requires a two-component Lyapunov morphism, with one component tracking time-to-next-impact and the other tracking an energy envelope that persists across transitions.

[*]Joint work with Aaron Ames.

The framework also surfaces a structural assumption implicit in all prior work on hybrid Lyapunov theory, which we call the decoupling assumption: the condition that the discrete and continuous components of the Lyapunov estimate are independent of one another. Making it explicit reveals what a more general hybrid Lyapunov theory would require. This work [3] is a first instantiation of the categorical Lyapunov framework of [2, 1] to hybrid systems, and the choice of base category, endofunctor, and system encoding are each modifiable, leaving substantial room for further development.

References

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