Monoidal categories graded by partial commutative monoids

EXTENDED ABSTRACT*

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Parallel composition of processes is often subject to constraints: programs accessing shared memory may only run in parallel safely if they operate on disjoint memory locations, or processes consuming a bounded resource may only run in parallel if their combined usage is within the bound. Effectful categories [7, 6, 8] axiomatize an instance of this phenomenon in programming language semantics: effectful computations may be combined in parallel with central computations, but not with each other. Monoidal categories axiomatize the unrestricted case, in which the globally defined monoidal product allows all processes to be put in parallel. This work introduces *monoidal categories graded by partial commutative monoids* (PCMs) to axiomatize the structure common to all of these situations.

A PCM $(E,\oplus,0)$ is a set E equipped with a unit element 0, and a partially defined binary operation $\oplus$, which is unital and associative up to Kleene equality. Every PCM has an extension preorder on its elements defined by $a \leqslant b \iff \exists c$ with $a \oplus c = b$.

An $(E,\oplus,0)$*-graded monoidal category* $\mathbb{C}$ comprises a monoid of objects $(\mathbb{C}_{\text{obj}},\otimes,I)$, a family of categories $\{\mathbb{C}_a\}_{a\in E}$ with objects $\mathbb{C}_{\text{obj}}$, indexed by elements of E (grades), equipped with identity-on-objects regrading functors $(-)_a^b : \mathbb{C}_a \to \mathbb{C}_b$ whenever $a \leqslant b$ in the extension preorder of E, and with monoidal product operations

$$(\otimes)_{a,b} : \mathbb{C}_a(X;Y) \times \mathbb{C}_b(X';Y') \to \mathbb{C}_{a\oplus b}(X \otimes X'; Y \otimes Y')$$

which exist only when $a \oplus b$ is defined. The key axioms are functoriality of regrading, compatibility of regrading with monoidal products, and an interchange law. Full details may be found in our preprint [2].

Examples

- Grading by the singleton PCM $\mathbf{1}$ precisely recovers strict monoidal categories.
- Grading by the two element PCM $\mathbf{2} = \{0,1\}$ with operation $0 \oplus 0 = 0$; $0 \oplus 1 = 1 \oplus 0 = 1$; and $1 \oplus 1$ *undefined*, precisely recovers effectful categories: $\mathbb{C}_0$ is monoidal, $\mathbb{C}_1$ is premonoidal, and the regrading $\mathbb{C}_0 \to \mathbb{C}_1$ is strict premonoidal, and has central image.
- Grading by a powerset PCM $\mathscr{P}(D)$ with the partial operation given by union on disjoint subsets, models non-interfering parallelism: morphisms are graded by the set of *devices* $X \in \mathscr{P}(D)$ that they use (such as memory locations), and their monoidal product is defined only when they use disjoint device sets. This generalizes the previous example, since $\mathbf{2} \cong \mathscr{P}(\mathbf{1})$.
- Grading by an interval $[0,r]$ with bounded addition models bounded resource usage: the monoidal product of morphisms is defined only when the sum of their grades does not exceed the bound r.

*This is an extended abstract of the preprint [2].

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- Consider the PCM $\mathsf{RW} := (\mathscr{P}(L)^2, \oplus, (\varnothing, \varnothing))$ where L is a set of memory locations, and

$$(R_1, W_1) \oplus (R_2, W_2) := \begin{cases} (R_1 \cup R_2, W_1 \cup W_2) & \text{if } W_1 \cap (R_2 \cup W_2) = \varnothing = W_2 \cap (R_1 \cup W_1) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

  Then morphisms in a monoidal category graded by RW have grade (R, W) indicating which memory locations they may read and write, and their monoidal product is only defined when writes do not conflict.

Results Our main results concerning E-graded monoidal categories are as follows.

- *Effectful categories are* **2***-graded monoidal categories.* As mentioned above, grading by an appropriate two element PCM recovers effectful categories, and this extends to an isomorphism of categories. This is a corollary of some general facts: every category $\mathbb{C}_a$ in an E-graded monoidal category has a canonical premonoidal structure, and whenever $a \in E$ is idempotent, $a \oplus a = a$, $\mathbb{C}_a$ is moreover monoidal.
- *Effectful categories are a coreflective subcategory of graded monoidal categories.* $\mathsf{Eff} \cong \mathbf{2}\text{-}\mathsf{GradMon}$ is a coreflective subcategory of PCM-graded monoidal categories whose grading PCM has a top element. The coreflector functorially "squashes" an E-graded monoidal category to a **2**-graded one by collapsing all non-zero grades to the grade 1.
- *Cartesian structure recovers Freyd categories.* We define symmetric and cartesian structure for PCM-graded monoidal categories and prove that cartesian **2**-graded monoidal categories are isomorphic to Freyd categories [6], with the coreflection restricting accordingly.
- *Global categories from graded monoidal categories.* We show that when the extension preorder of the grading PCM has binary joins, the local categories $\{\mathbb{C}_a\}_{a \in E}$ assemble into a single category with sequential composition $f \fatsemi g := f_a^{a \vee b} \fatsemi g_b^{a \vee b}$. For instance, in a $\mathscr{P}(D)$-graded monoidal category, this gives sequential composition of programs that use different sets of devices, via regrading to their union. A similar construction holds when the extension preorder is merely directed.
- *PCM-graded monoidal categories are monoids.* Viewing a PCM as a thin promonoidal category [1], we obtain a convolution monoidal structure on presheaves which, paired with the pointwise cartesian product, forms a duoidal category. Using results of Heunen and Sigal [4], we characterize E-graded monoidal categories as monoids in a particular monoidal category of lax monoidal functors.

Related work Graded monads [5] and graded Freyd categories [3] equip morphisms with grades that combine under *sequential* composition, tracking accumulated effects or costs. In PCM-graded monoidal categories, sequential composition preserves grades, and grades instead combine under the *monoidal product*, but only when their combination is defined. This makes PCM-graded monoidal categories suited to modelling ambient resources such as memory locations or bandwidth, rather than sequentially accumulated information. Sarkis and Zanasi [9] study a notion of graded monoidal category in which grades combine (totally) under both sequential and parallel composition.

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