Picturing causality and localisability on non-factor subsystems

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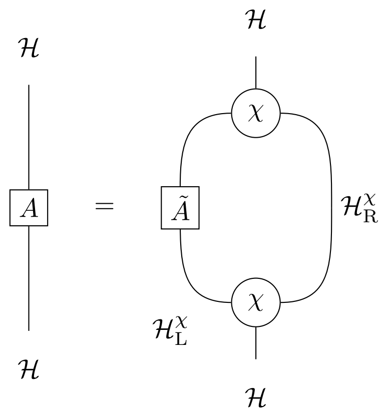
We propose to present a way of representing general quantum subsystems in a circuit model (using the language of symmetric monoidal categories) and use it to obtain a causal decomposition result (i.e. an equivalence between the signalling properties of an evolution and a compositional structure) for quantum channels that are one-way signalling between general quantum subsystems.

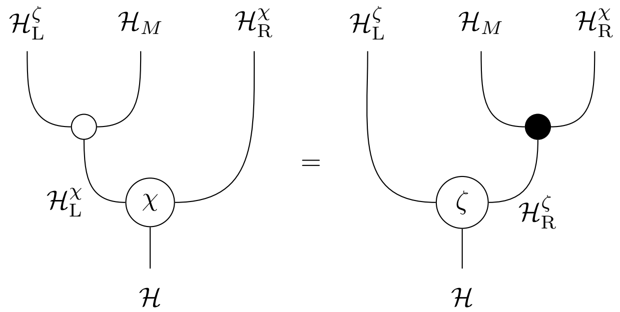
1 Picturing non-factor subsystems

Our approach consists in representing subsystems of a (finite-dimensional) Hilbert space $\mathcal{H}$ by isometries $\chi : \mathcal{H} \to \mathcal{H}_{\mathrm{L}}^{\chi} \otimes \mathcal{H}_{\mathrm{R}}^{\chi}$ which we call splitting maps. A splitting map χ induces a notion of *χ-local operators* which are the elements $A \in \mathcal{L}(\mathcal{H})$ that can be seen as acting locally with respect to the left (or the right) of χ, i.e. such that $A = \chi\dagger(\tilde{A} \otimes \mathbb{1})\chi$ (resp. $A = \chi\dagger(\mathbb{1} \otimes \tilde{A})\chi$).



(1)

We prove that the set of such local operators has the structure of an operator system and that its subset of elements (that we call *strictly χ-local*), whose on-site $\tilde{A}$ commutes with $\chi\chi^{\dagger}$, form a (finite-dimensional) von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$, allowing us to connect splitting maps to the usual algebraic definition of subsystem. In order to phrase in our formalism that a subsystem (extracted by the left branch of ζ) is included in another (extracted by the left branch of χ), we define a notion of inclusion between splitting maps, termed comprehension and written $\sqsubseteq$, represented by the following equation:

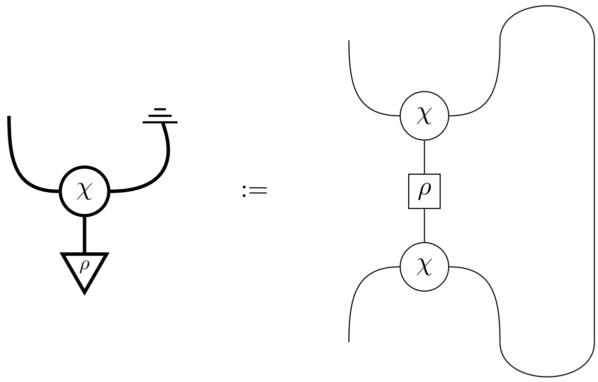


(2)

We show that given a fixed Hilbert space $\mathcal{H}$, comprehension is a preorder relation on the set $\text{split}(\mathcal{H})$ of splitting maps with $\mathcal{H}$ as their base space and that this preordered set $(\text{split}(\mathcal{H}), \sqsubseteq)$ is categorically equivalent to the preordered set $(\text{vnalg}(\mathcal{H}), \subseteq)$ of the von Neumann subalgebras of $\mathcal{L}(\mathcal{H})$ equipped with their usual inclusion. And this equivalence is witnessed through strictly local operators and the canonical representation theorem of finite dimensional von Neumann algebras. We then use this equivalence to prove the following causal decomposition result for quantum channels that are one-way signalling between general non-factor subsystems.

2 Causality between non-factor subsystems

We begin by extending the notion of splitting map that we had defined in pure quantum theory to mixed quantum theory by doubling the isometry, just as it would be done for a unitary that we would like to see as a channel [1]. This allows us to define the notion of trace over a splitting map as:



(3)

where the right part is an unfolding (in string diagrams for the category of finite-dimensional Hilbert spaces) of the left, which is a string diagram for the category of completely positive trace-preserving maps. We show that this notion of trace over over a splitting map is equal (up to an isometry due to the choice of representation) to the trace over the algebra [2] of its strictly local operators. We are then able to express when a quantum channel $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is Schrödinger semi-causal (non-signalling) from a subsystem $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ of the input to a subsystem $\mathcal{B} \subseteq \mathcal{L}(\mathcal{K})$ of the output, i.e. when it satisfies the left-hand side equality of Figure 1.

We show that this definition generalises both the traditional definition of non-signalling of a quantum channel between factor subsystem the operational definition of non-signalling (or Heisenberg semi-causality) of a unitary between non-factor subsystems. Finally, we prove a causal decomposition theorem for these quantum channels that are non-signalling between non-factor quantum subsystems, generalising the result of [3].

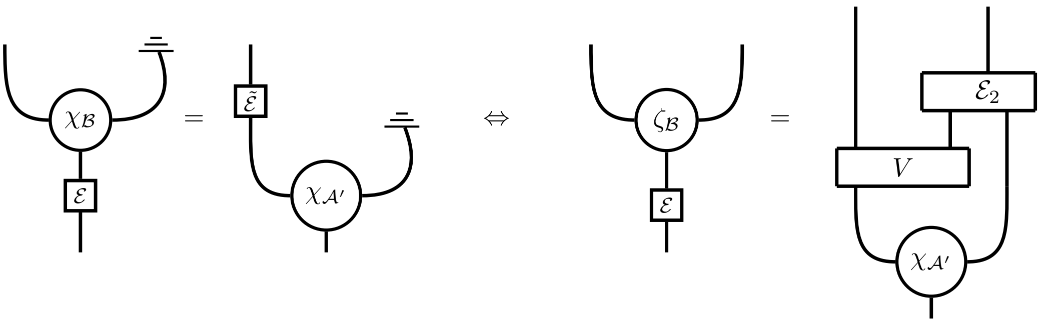


Figure 1: A causal decomposition result for channels that are non-signalling from a general quantum subsystem to another.

References

[1] Bob Coecke and Aleks Kissinger. "Picturing quantum processes: A first course in quantum theory and diagrammatic reasoning". Cambridge University Press. (2017).

[2] Giulio Chiribella. "Agents, subsystems, and the conservation of information". Entropy **20**, 358 (2018).

[3] T Eggeling, D Schlingemann, and R. F Werner. "Semicausal operations are semilocalizable". Europhysics Letters (EPL) **57**, 782–788 (2002).