Algorithmic and Extremal Obstructions Through the Language of Cohomology (extended abstract)

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We model decision problems as presheaves that assign sets of certificates to input instances and we show how to use *presheaf* Čech cohomology to capture the precise ways in which local solutions fail to patch into global ones. Applied to problems like VertexCover, CycleCover, and OddCycleTransversal, our framework exposes emergent phenomena, such as hidden cycles or the inflation of small, local solutions. This approach not only rephrases classical results like König's Theorem in cohomological terms, but also reveals how to systematically account for failures of compositionality. Although our main focus is on presheaves of sets, the methods generalize naturally to Abelian presheaves, suggesting a rich interplay between graph theory, cohomology, and complexity. This work represents a first step toward a systematic, sheaf-theoretic theory of algorithmic structure and related obstructions.

1 Obstructions to compositionality

Dynamic programming is a common approach in algorithmics and the idea is as follows: to find solutions to computational problem on an input G, one works locally on small parts thereof and then finds ways of patching these local solutions together into global solutions on G. A key step in designing these kinds of algorithms is that of identifying how local solutions may fail to patch together and how to keep track of this information. This is what we mean by *obstructions to compositionality*.

We define problems as presheaves and we use the concept of sheaf to encode the property of how compatible local data can be assembled into global data. It is a known result [1] that problems which can be described as sheaves can be solved with FPT-time algorithms. This talk (based on [2]) aims to use categorical and cohomological tools to investigate when problems fail to display compositionality and how to account for these obstructions. As case studies, we work with three examples on graphs, VertexCover, CycleCover, and OddCycleTransversal, and describe them as presheaves as follows.

Given $k \in \mathbb{N}$ and G a graph, we can define functors $\mathcal{V}_{\leq k}, \mathcal{C}_{\leq k}, \mathcal{B}_{\leq k} \colon \mathsf{Sub}(G)^{\mathsf{op}} \to \mathsf{Set}$ given by $\mathcal{V}_{\leq k}(H) = \{A \subseteq V(H) : H - A \text{ is edgeless and } |A| \leq k\}$, $\mathcal{C}_{\leq k}(H) = \{A \subseteq V(H) : H - A \text{ doesn't have cycles and } |A| \leq k\}$ and $\mathcal{B}_{\leq k}(H) = \{A \subseteq V(H) : H - A \text{ doesn't have odd cycles and } |A| \leq k\}$ for every subgraph $H \subseteq G$, and we define the action on the arrows (which in $\mathsf{Sub}(G)$ are the inclusions) by the restriction.

Given a category equipped with a notion of covers for its objects, one can speak of sheaves: these are presheaves for which there is a bijective correspondence between global data and pairwise agreeing data assignments on the parts of the cover. In our case, where the category is $\mathsf{Sub}(G)$ and the cover is subgraphs $H_i \subseteq H$ such that $H = \bigcup_{i \in I} H_i$, this means that the solutions $\{A_i \in F(H_i)\}_{i \in I}$ that agree on the restrictions $A_i \restriction_{H_i \cap H_j} = A_j \restriction_{H_i \cap H_j}$ are in a bijective correspondence with $F(H)$, for all $H \in \mathsf{Sub}(G)$. In the talk, we will explain how to use cohomology to measure the failure of being a sheaf.

1.1. Proposition. $\mathcal{V}_{\leq k}, \mathcal{C}_{\leq k}$ and $\mathcal{B}_{\leq k}$ are separated presheaves, but not sheaves.

To define cohomology, we need to work with presheaves valued in abelian categories. We do this for our examples by free abelianization. Note that here we are working with *presheaf* cohomology, not the standard sheaf cohomology. The two agree for all cohomology groups H^n when $n \geq 1$, however, they *do not agree on* H^0. In our case, H^0 is defined in terms of an extended Čech nerve

$$X \xleftarrow{d_0} \amalg_{i \in I} H_i \underset{d_{1,1}}{\overset{d_{1,0}}{\leftleftarrows\!\!\rightarrow}} \amalg_{i,j \in I} H_i \cap H_j \qquad\qquad F(X) \xrightarrow{\delta^{-1}_{\mathcal{U}}} \prod_i F(H_i) \xrightarrow{\delta^{0}_{\mathcal{U}}} \prod_{i,j} F(H_i \cap H_j)$$

where the coboundary maps are given by $\delta^{-1}_{\mathcal{U}} := F(d_0)$ and $\delta^{0}_{\mathcal{U}} := F(d_{1,0}) - F(d_{1,1})$.

1.2. Definition. The **zeroth Čech cohomology** of a presheaf $F\colon \mathsf{Sub}(G)^{\mathsf{op}} \to \mathsf{Ab}$ on an object X with cover $\mathcal{U}$ is defined as $H^0(x, \mathcal{U}, F) := \mathsf{coker}\big(\mathsf{im}(\delta^{-1}) \to \mathsf{ker}(\delta^0)\big)$.

The generators of H^0 indicate the local sections that, despite agreeing on the intersections, fail to be lifted to global sections. In short, the zeroth presheaf Čech cohomology — in contrast to sheaf cohomology, where H^0 describes global sections — can be thought of as a measure of the failure of being a sheaf.

Although sheafification is a canonical way to construct a sheaf given a presheaf, this doesn't work on our algorithmic perspective. For example, the sheafification of $\mathcal{V}_{\leq k}$ is the sheaf $\mathcal{V}(H) := \{A \subseteq V(H) : H - A \text{ is edgeless}\}$, without the size restriction, which does not encode an interesting decision problem, since all graphs have a vertex cover: the entire vertex set. Nonetheless, sheafification turns up in our following characterization of presheaf Čech cohomology, a result that we shall use to characterize the failures of compositionality that we are interested in.

1.3. Theorem. Suppose F is a separated Abelian presheaf. Then, letting $\eta\colon F \Rightarrow F^+$ be the unit of the adjunction given by sheafification, one has that $H^0(-, F) = \mathsf{coker}(F \xrightarrow{\eta_F} F^+)$.

Applied to our examples, for instance, the zeroth *presheaf* Čech cohomology group accounts for all *emergent obstructions* such as local small solutions becoming too large when joined together or the emergence of (odd) cycles that are not visible in the small constituent subgraphs comprising a (odd) cycle cover, as stated in the following result.

1.4. Proposition. For $k \geq 2$, $H^0(X, \mathcal{V}_{\leq k}) = \mathbb{Z}[\{A \subseteq X : H - A \text{ is edgeless and } |A| > k\}]$, $H^0(X, \mathscr{C}_{\leq k}) = \mathbb{Z}[\{A \subseteq X : |A| > k \text{ or } X - A \text{ has a cycle}\}]$ and $H^0(X, \mathscr{B}_{\leq k}) = \mathbb{Z}[\{A \subseteq X : |A| > k \text{ or } X - A \text{ has an odd cycle}\}]$.

2 Identifying other kinds of obstruction using presheaf cohomology

Besides considering the obstructions to "algorithmic compositionality" (i.e. obstructions to being a sheaf), we can also use zeroth presheaf cohomology to talk about obstructions to having *any* solution at all.

2.1. Theorem. There exists a covariant functor $\mathfrak{M}$, called the **model collecting functor**, that maps any presheaf $F\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$, where C is a category with pullbacks, to a flasque presheaf $\mathfrak{M}F$. Moreover, if $\mathsf{C} = \mathsf{Sub}(G)$ and the complete graph with two vertices has a solution for F, then $H^0(X, \mathfrak{M}F) = \mathbb{Z}[\{X' \subseteq X : F(X') = \emptyset\}]$ and we have that $FX \neq \emptyset$ iff $H^0(X, \mathfrak{M}F) = 0$.

The above result tells us we can use cohomology to decide if a graph X has a solution for a certain problem F. We can use this to restate a classic result in terms of the zeroth presheaf cohomology of the model collecting functor. Although we don't provide a new algebraic proof, the following result points to the fact that cohomological methods provide a possible avenue towards stating and proving results in extremal graph theory.

2.2. Theorem (König's Theorem, Cohomologically)**.** If X is a bipartite graph, then $H^0(X, \mathfrak{M}\mathcal{V}_{\leq k}) = \{\text{matchings of } X \text{ of size greater than } k\}$.

By adapting Čech cohomology to *presheaves*, our work illustrates how existing cohomological methods can be meaningfully applied to classical algorithmic problems. These observations invite further exploration at the intersection of topology, category theory, and algorithm design.

References

[1] Althaus, E., Bumpus, B. M., Fairbanks, J., and Rosiak, D. Compositional algorithms on compositional data: Deciding sheaves on presheaves. *arXiv preprint arXiv:2302.05575* (2023).

[2] Azevedo, A. B., Bumpus, B. M., Capucci, M., Fairbanks, J., and Rosiak, D. Algorithmic and extremal obstructions through the language of cohomology. *arXiv preprint arXiv:arXiv:2407.03488* (2025).