

Measuring Univalence and Its Failure in Homotopy Type Theory via Synthetic Cohomology

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We study how a chosen universe or type family in homotopy type theory can (fail to) satisfy univalence by means of synthetic cohomology. For each type family, we define a defect space from the map that compares equality of codes with equivalence of the corresponding decoded types, and we use synthetic cohomology of this space to build invariants. These invariants vanish for univalent families, completely detect triviality in truncated defect spaces, and can be computed in several concrete examples. We show that the construction extends to Tarski universes and captures information that ordinary cohomology of the fibers does not detect.

1 Introduction

Homotopy type theory (HoTT) treats types as spaces and identity proofs as paths. A chosen type family (U, El) is univalent when equality of codes agrees with equivalence of the corresponding decoded types [14, 1, 8, 6, 15]. At the same time, higher inductive methods and synthetic cohomology make it possible to carry out substantial homotopy-theoretic and cohomological constructions internally in HoTT [11, 13, 7, 3, 10, 5, 4, 2, 9, 12].

Univalence is usually presented as a sharp structural property: for a given family, equality of codes either does or does not coincide with equivalence of decoded types. But concrete universes and type families are rarely encountered only as abstract yes/no objects. They come with specific coding choices, closure operations, and computational presentations. When univalence fails, a bare negative answer does not say where the failure lives, whether it is stable under equivalence of presentations, or whether it can be detected internally by standard homotopy-theoretic tools. A useful invariant should therefore do more than announce non-univalence: it should turn the discrepancy itself into geometric data that can be compared, computed, and transported across settings.

This paper asks for an invariant of a specified type family that measures the gap between these two notions of sameness:

Can one attach a canonical cohomological invariant to a specified type family (U, El) that vanishes under univalence and yields explicit nontrivial classes for non-univalent presentations?

Our answer is yes. For every chosen family

$$\mathcal{U} = (U, \text{El})$$

we consider the canonical map

$$\iota_{\mathcal{U}} : U \rightarrow \text{Eq}(\mathcal{U}), \quad u \mapsto (u, u, \text{id}_{\text{El}(u)}),$$

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from codes to self-equivalences of decoded types, and define the *univalence defect space*

$$D(\mathcal{U}) := \text{Cofib}(\iota_{\mathcal{U}}).$$

Intuitively, $U \simeq P(U)$ remembers only the equivalences forced by literal equality of codes, whereas $\text{Eq}(\mathcal{U})$ remembers all equivalences visible after decoding. The cofiber $D(\mathcal{U})$ is the quotient of the second by the first: it collapses the tautological copy of the code space inside the equivalence space and retains the residual geometry that equality does not explain. Thus the failure of univalence is converted into an actual pointed type, so ordinary synthetic cohomology can be applied to it. If \mathcal{U} is univalent, then $\iota_{\mathcal{U}}$ is an equivalence, so $D(\mathcal{U})$ is contractible. Applying synthetic cohomology to this pointed type yields the canonical invariants

$$H_{\text{UD}}^n(\mathcal{U}; A) := \tilde{H}^n(D(\mathcal{U}); A), \quad H_{\text{UD,na}}^1(\mathcal{U}; G) := [D(\mathcal{U}), BG]_*.$$

The construction sits between two established strands of work. One is the study of universes, univalence, and their semantic or computational realizations in simplicial and cubical settings [8, 6, 15]. The other is the internal development of synthetic cohomology and its explicit calculations in HoTT and cubical type theory [10, 5, 4, 2, 9, 12]. What is missing from the second story is a canonical space built from the very comparison that univalence discusses. The defect cofiber provides exactly that space. This is significant in three ways. Conceptually, it refines univalence from a binary predicate to a graded obstruction theory whose different degrees isolate different layers of failure. Methodologically, it feeds the comparison map for a family directly into the existing machinery of synthetic cohomology, so universes and type families themselves become objects of cohomological computation. Semantically, it yields invariants that distinguish different chosen universe presentations even inside the same ambient model, showing that the placement of the code space inside the equivalence space is meaningful structure rather than bookkeeping.

The main results are as follows.

- (1) For truncated defect spaces, the package

$$H_{\text{UD}}^0(\mathcal{U}; \mathbb{Z}), \quad H_{\text{UD,na}}^1(\mathcal{U}; G), \quad H_{\text{UD}}^n(\mathcal{U}; A) \quad (n \geq 2)$$

detects contractibility of $D(\mathcal{U})$. In other words, once the defect space has finite Postnikov length, these invariants recover whether any defect remains at all.

- (2) For the constant unit family

$$\mathcal{C}_B := (B, \underline{1}),$$

one has

$$D(\mathcal{C}_B) \simeq \text{Cofib}(\Delta_B).$$

This yields explicit examples: spheres give nontrivial classes in prescribed degrees, finite discrete types concentrate in degree 0, and bouquets and tori exhibit richer rank patterns.

- (3) If \mathcal{U} is a Tarski universe containing a code for 1, then the thickening

$$\mathcal{U} \boxtimes B := (U \times B, \text{El} \circ \text{pr}_1)$$

preserves the specified algebraic structure and has defect space containing $D(\mathcal{C}_B)$ as a retract. Hence the explicit constant-family classes can be realized inside Tarski universes.

- (4) These invariants vanish on distinguished univalent universes, are nontrivial on suitable explicit thickenings, and are not determined by the ordinary cohomology of the decoded fibers alone.

Taken together, these statements show that defect cohomology is not just an obstruction that occasionally witnesses failure. It is computable, stable under equivalence of families, transportable into structured settings, and sensitive to global features of a universe presentation that fiberwise invariants do not see.

The paper is organized as follows. Section 2 introduces the defect space and its basic functoriality. Section 3 gives the detection theorem for truncated defect spaces. Sections 4 and 5 compute explicit classes and realize them inside Tarski universes. Sections 6 and 7 extract the structured-model comparison and the non-reducibility to fiberwise cohomology. The appendix supplies the longer proofs and calculations in the same order.

2 Defect spaces and defect cohomology

We work internally in a fixed ambient form of HoTT with identity types, dependent sums and products, enough higher inductive structure to form pushouts and cofibers, truncations, and Eilenberg-MacLane types [14, 10, 6, 15]. The basic object in this paper is a specified type family $\mathcal{U} = (U, \text{El})$. We reserve the word *universe* for the structured cases considered later. The defect invariant is attached to that chosen datum itself; in particular, for a structured model it is an invariant of the chosen distinguished universe.

We write $[X, Y]_*$ for pointed homotopy classes of pointed maps and point every cofiber at its cone point. For $n \geq -2$, $\tau_{\leq n}X$ denotes the n -truncation of X . For an abelian group A and $n \geq 0$, let $K(A, n)$ denote a pointed Eilenberg–MacLane type (for a construction in HoTT, see [10]), and define reduced cohomology by

$$\tilde{H}^n(X; A) := [X, K(A, n)]_*.$$

For a general group G , we write BG for a pointed connected 1-type with loop space G . We suppress universe levels throughout; if size issues matter, all constructions may be read one universe higher.

Definition 2.1. A *small type family* (or simply a *family*) is a pair $\mathcal{U} = (U, \text{El})$ with $\text{El} : U \rightarrow \text{Type}$. We regard elements of U as *codes* and $\text{El}(u)$ as the type decoded by the code u .

For every family $\mathcal{U} = (U, \text{El})$, there are two associated types:

$$P(U) := \sum_{u, v: U} (u = v), \quad \text{Eq}(\mathcal{U}) := \sum_{u, v: U} (\text{El}(u) \simeq \text{El}(v)).$$

Transport in El yields the canonical comparison map

$$\text{idtoeq}_{\mathcal{U}} : P(U) \rightarrow \text{Eq}(\mathcal{U}).$$

Path induction identifies $P(U)$ with U through the map

$$\tau_U : U \rightarrow P(U), \quad \tau_U(u) := (u, u, \text{refl}_u).$$

The type $P(U)$ records sameness coming from literal equality of codes, while $\text{Eq}(\mathcal{U})$ records sameness visible only after decoding. Univalence asserts that these two spaces agree. The defect construction starts from the idea that the discrepancy between them should be studied in the same way one studies the difference between a subspace and its ambient space.

Definition 2.2. The *defect map* of \mathcal{U} is

$$\iota_{\mathcal{U}} : U \rightarrow \text{Eq}(\mathcal{U}), \quad u \mapsto (u, u, \text{id}_{\text{El}(u)}),$$

equivalently $\iota_{\mathcal{U}} = \text{id}_{\text{toeq}_{\mathcal{U}}} \circ \tau_U$. The *univalence defect space* of \mathcal{U} is the pointed type

$$D(\mathcal{U}) := \text{Cofib}(\iota_{\mathcal{U}}).$$

Proposition 2.3. *A family \mathcal{U} is univalent if and only if $\iota_{\mathcal{U}}$ is an equivalence. Consequently, univalence implies that $D(\mathcal{U})$ is contractible.*

Remark 2.4. The map $\iota_{\mathcal{U}}$ singles out the diagonal copy of the code space inside $\text{Eq}(\mathcal{U})$: for each code u , it remembers only the identity self-equivalence of $\text{El}(u)$. Elements of $\text{Eq}(\mathcal{U})$ may also encode equivalences between distinct codes or nontrivial higher loops of autoequivalences, and these are precisely the kinds of data that survive after passing to the cofiber. Thus $D(\mathcal{U})$ should be pictured as the residue obtained by collapsing the “obvious” equivalences and asking what geometry of equivalences remains. For a general map f , however, contractibility of $\text{Cofib}(f)$ does not imply that f is an equivalence, so the defect space is best viewed first as an obstruction object rather than as a complete reformulation of univalence.

Definition 2.5. For an abelian group A and $n \geq 0$, the *n -th univalence defect cohomology group* of \mathcal{U} is

$$H_{\text{UD}}^n(\mathcal{U}; A) := \tilde{H}^n(D(\mathcal{U}); A) = [D(\mathcal{U}), K(A, n)]_*.$$

For an arbitrary group G , the *nonabelian degree-one defect invariant* is the pointed set

$$H_{\text{UD,na}}^1(\mathcal{U}; G) := [D(\mathcal{U}), BG]_*.$$

Proposition 2.6. *For every abelian group A and every $n \geq 0$, there is a canonical identification*

$$H_{\text{UD}}^n(\mathcal{U}; A) \cong H^n(\text{Eq}(\mathcal{U}), U; A),$$

where the right-hand side is relative cohomology with respect to $\iota_{\mathcal{U}} : U \rightarrow \text{Eq}(\mathcal{U})$.

This relative description is often the most useful computational viewpoint. A defect class is simply a cohomology class on the full equivalence space that becomes trivial when restricted to the diagonal copy of codes. In concrete calculations, one first identifies $\text{Eq}(\mathcal{U})$, then asks which classes vanish on the image of $\iota_{\mathcal{U}}$.

Definition 2.7. Let $\mathcal{U} = (U, \text{El})$ and $\mathcal{V} = (V, \text{El}')$ be small type families. A *morphism of type families*

$$(F, \phi) : \mathcal{U} \rightarrow \mathcal{V}$$

consists of a map $F : U \rightarrow V$ together with a dependent family of equivalences

$$\phi : \prod_{u:U} (\text{El}(u) \simeq \text{El}'(F(u))),$$

and we write $\phi_u := \phi(u)$.

Such a morphism induces a map

$$\text{Eq}(F, \phi)(u, v, e) := (F(u), F(v), \phi_v \circ e \circ \phi_u^{-1})$$

on equivalence spaces.

Proposition 2.8. *Every morphism of type families $(F, \phi) : \mathcal{U} \rightarrow \mathcal{V}$ induces a canonical pointed map*

$$D(F, \phi) : D(\mathcal{U}) \rightarrow D(\mathcal{V}).$$

These maps are functorial with respect to identities and composition. Consequently, for every abelian group A , every group G , and every $n \geq 0$, the assignments $H_{\text{UD}}^n(-; A)$ and $H_{\text{UD,na}}^1(-; G)$ are contravariantly functorial.

Definition 2.9. A morphism of type families

$$(F, \phi) : \mathcal{U} \rightarrow \mathcal{V}$$

is an *equivalence of type families* if its underlying map

$$F : U \rightarrow V$$

is an equivalence.

Remark 2.10. This is the homotopically natural notion of sameness for families in the present setting. Definition 2.7 already requires equivalences on all decoded fibers, so the additional condition is only that the code map F be an equivalence. A stricter two-sided inverse condition is unnecessary for the homotopy-invariant constructions used below.

Proposition 2.11. *If $(F, \phi) : \mathcal{U} \rightarrow \mathcal{V}$ is an equivalence of type families, then the induced map*

$$\text{Eq}(F, \phi) : \text{Eq}(\mathcal{U}) \rightarrow \text{Eq}(\mathcal{V})$$

is an equivalence.

The next corollary gives the invariance statement used later: defect cohomology is unchanged under equivalence of type families.

Corollary 2.12. *If $(F, \phi) : \mathcal{U} \rightarrow \mathcal{V}$ is an equivalence of type families, then $D(F, \phi)$ is an equivalence of pointed types. In particular, for every abelian group A , every group G , and every $n \geq 0$, there are canonical isomorphisms*

$$H_{\text{UD}}^n(\mathcal{U}; A) \cong H_{\text{UD}}^n(\mathcal{V}; A), \quad H_{\text{UD,na}}^1(\mathcal{U}; G) \cong H_{\text{UD,na}}^1(\mathcal{V}; G).$$

In particular, defect cohomology does not depend on inessential choices of coding syntax once the whole family is replaced by an equivalent one. What it remembers are genuine geometric features of the presentation, not artifacts of notation.

Theorem 2.13. *If \mathcal{U} is univalent, then for every abelian group A , every group G , and every $n \geq 0$,*

$$H_{\text{UD}}^n(\mathcal{U}; A) = 0 \quad \text{and} \quad H_{\text{UD,na}}^1(\mathcal{U}; G) = *.$$

Proposition 2.14. *Let G be a group.*

(i) *The assignment*

$$\mathcal{U} \longmapsto H_{\text{UD,na}}^1(\mathcal{U}; G)$$

is contravariantly functorial with respect to morphisms of type families.

(ii) *If G is abelian, then there is a canonical identification*

$$H_{\text{UD,na}}^1(\mathcal{U}; G) \cong H_{\text{UD}}^1(\mathcal{U}; G).$$

(iii) If \mathcal{U} is univalent, then

$$H_{\text{UD,na}}^1(\mathcal{U}; G)$$

is the one-point set.

The role of $H_{\text{UD,na}}^1(\mathcal{U}; G)$ is that it captures the first Postnikov-level obstruction carried by the defect space. When $D(\mathcal{U})$ is connected but not simply connected, the canonical map to the 1-truncation

$$D(\mathcal{U}) \rightarrow \tau_{\leq 1} D(\mathcal{U}) \simeq B\pi_1(D(\mathcal{U}))$$

determines a class in

$$H_{\text{UD,na}}^1(\mathcal{U}; \pi_1(D(\mathcal{U}))).$$

This is why the nonabelian degree-one invariant is the correct recipient for the first defect class before higher abelian cohomology enters. In heuristic terms, the package

$$H_{\text{UD}}^0, \quad H_{\text{UD,na}}^1, \quad H_{\text{UD}}^n \quad (n \geq 2)$$

tracks the component, fundamental-group, and higher Postnikov layers of the defect space in exactly the order one would expect from homotopy theory.

Thus defect cohomology is a canonical obstruction to non-univalence. Its role is not to replace ordinary synthetic cohomology, but to supply a canonical pointed type on which the ordinary cohomology theory is evaluated.

3 Detection on truncated defect spaces

Contractibility of $D(\mathcal{U})$ is weaker than univalence in general, but on truncated defect spaces, the cohomological package above completely detects whether the defect space itself is trivial. This should be read as a synthetic Postnikov statement: H_{UD}^0 detects whether any non-basepoint components remain, $H_{\text{UD,na}}^1$ detects residual fundamental-group data, and the higher groups detect the successive abelian layers. Once the defect space is truncated, there are no further homotopy layers in which a hidden defect could live, so vanishing of the whole package forces the entire defect space to collapse.

Theorem 3.1. *Let $\mathcal{U} = (U, \text{El})$ be a small type family, and assume that*

$$D(\mathcal{U}) = \text{Cofib}(\iota_{\mathcal{U}})$$

is N -truncated for some finite $N \geq 0$. Then the following are equivalent:

- (i) $D(\mathcal{U})$ is contractible.
- (ii) The following conditions hold:
 - (a) $H_{\text{UD}}^0(\mathcal{U}; \mathbb{Z}) = 0$.
 - (b) For every group G ,

$$H_{\text{UD,na}}^1(\mathcal{U}; G)$$

is the one-point set.

- (c) For every abelian group A and every n with $2 \leq n \leq N$,

$$H_{\text{UD}}^n(\mathcal{U}; A) = 0.$$

Corollary 3.2. *Assume that $D(\mathcal{U})$ is N -truncated. If any one of the invariants*

$$H_{\text{UD}}^0(\mathcal{U}; \mathbb{Z}), \quad H_{\text{UD,na}}^1(\mathcal{U}; G), \quad H_{\text{UD}}^n(\mathcal{U}; A) \quad (2 \leq n \leq N)$$

is nontrivial for some choice of coefficients, then \mathcal{U} is not univalent.

Thus, on truncated defect spaces, the invariant has a clean role: any nontrivial defect class obstructs univalence, and vanishing of the full package is equivalent to contractibility of $D(\mathcal{U})$. What it does *not* do by itself is upgrade contractible defect to univalence; that requires extra hypotheses on the family or on the map $\iota_{\mathcal{U}}$.

4 Constant unit families

Let B be any type, and write

$$\mathcal{C}_B := (B, \underline{1})$$

for the type family constant at the unit type. This is the cleanest test case: every decoded fiber is contractible, so any nontrivial defect class must come entirely from the geometry of codes and equivalences. There is a simple picture to keep in mind. A point of the eventual space $B \times B$ will record a source code and a target code, the diagonal will record the pairs where the two codes literally agree, and the cofiber will collapse exactly that diagonal. Thus every class in this section measures the gap between “same code” and “possibly different codes,” not any complexity of the fiber 1.

Proposition 4.1. *For every type B , there is a canonical equivalence*

$$\text{Eq}(\mathcal{C}_B) \simeq B \times B.$$

Under this identification, the defect map of \mathcal{C}_B is the diagonal $\Delta_B : B \rightarrow B \times B$. Consequently,

$$D(\mathcal{C}_B) \simeq \text{Cofib}(\Delta_B)$$

and, for every abelian group A and every $n \geq 0$,

$$H_{\text{UD}}^n(\mathcal{C}_B; A) \cong H^n(B \times B, B; A),$$

where the relative group is taken with respect to Δ_B .

Thus the defect cohomology of \mathcal{C}_B reduces to ordinary relative cohomology of the diagonal inclusion $B \rightarrow B \times B$. Equivalently, it measures which cohomology classes on the full source-target code space $B \times B$ disappear when the two codes are forced to coincide along the diagonal. Once the defect space is identified, the calculations are classical.

We begin with spheres because they provide the basic witnesses used later in the thickening and separation arguments. Finite discrete types, bouquets of spheres, and tori then show three broader regimes: purely degree-zero behavior, two-step behavior with controlled multiplicities, and nonvanishing across a whole range of degrees. The longer proofs are deferred to Appendix C.

Theorem 4.2. *Let $m \geq 1$. Then*

$$H_{\text{UD}}^k(\mathcal{C}_{S^m}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = m \text{ or } k = 2m, \\ 0, & \text{otherwise.} \end{cases}$$

If $u \in H^m(S^m; \mathbb{Z}) \cong \mathbb{Z}$ is a generator, then a generator of $H_{\text{UD}}^m(\mathcal{C}_{S^m}; \mathbb{Z})$ is represented by

$$\text{pr}_1^*(u) - \text{pr}_2^*(u) \in H^m(S^m \times S^m; \mathbb{Z}).$$

The generator $\text{pr}_1^*(u) - \text{pr}_2^*(u)$ gives the basic geometric pattern that recurs throughout the examples: the two projections agree on the diagonal, so their difference vanishes there and defines a relative class, but away from the diagonal it records the failure of the left and right copies of the code space to coincide.

Proposition 4.3. For $S^0 = 2$,

$$H_{\text{UD}}^0(\mathcal{C}_{S^0}; \mathbb{Z}) \cong \mathbb{Z}^2, \quad H_{\text{UD}}^k(\mathcal{C}_{S^0}; \mathbb{Z}) = 0 \quad (k > 0).$$

At this point the basic sphere example is complete: S^m gives a single prescribed positive degree together with the top class, and S^0 captures the disconnected degree-zero boundary case. The next three computations are not needed for the minimal separation theorem, but they show that the same diagonal-cofiber method naturally produces richer rank patterns once the code space carries more combinatorial or multiplicative structure.

Proposition 4.4. Let F_n be a discrete type with exactly $n \geq 1$ points. Then

$$H_{\text{UD}}^0(\mathcal{C}_{F_n}; \mathbb{Z}) \cong \mathbb{Z}^{n(n-1)}, \quad H_{\text{UD}}^k(\mathcal{C}_{F_n}; \mathbb{Z}) = 0 \quad (k > 0).$$

For $n = 2$, this recovers Proposition 4.3.

Theorem 4.5. Let

$$W_{r,m} := \bigvee_{i=1}^r S^m$$

with $r \geq 1$ and $m \geq 1$. Then

$$H_{\text{UD}}^k(\mathcal{C}_{W_{r,m}}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^r, & k = m, \\ \mathbb{Z}^{r^2}, & k = 2m, \\ 0, & \text{otherwise.} \end{cases}$$

If u_1, \dots, u_r is the basis of $H^m(W_{r,m}; \mathbb{Z})$ dual to the sphere summands, then a basis of $H_{\text{UD}}^m(\mathcal{C}_{W_{r,m}}; \mathbb{Z})$ is represented by

$$\text{pr}_1^*(u_i) - \text{pr}_2^*(u_i) \quad (1 \leq i \leq r),$$

and a basis of $H_{\text{UD}}^{2m}(\mathcal{C}_{W_{r,m}}; \mathbb{Z})$ is represented by

$$\text{pr}_1^*(u_i) \smile \text{pr}_2^*(u_j) \quad (1 \leq i, j \leq r).$$

Theorem 4.6. Let

$$T^n := (S^1)^n$$

be the n -torus. Then, for every $0 \leq k \leq 2n$,

$$H_{\text{UD}}^k(\mathcal{C}_{T^n}; \mathbb{Z})$$

is free abelian of rank

$$\binom{2n}{k} - \binom{n}{k},$$

and $H_{\text{UD}}^k(\mathcal{C}_{T^n}; \mathbb{Z}) = 0$ for $k > 2n$. Equivalently,

$$H_{\text{UD}}^k(\mathcal{C}_{T^n}; \mathbb{Z}) \cong \ker(\Delta_{T^n}^* : H^k(T^n \times T^n; \mathbb{Z}) \rightarrow H^k(T^n; \mathbb{Z})).$$

In degree 1, if e_1, \dots, e_n is the standard basis of $H^1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n$, then a basis of $H_{\text{UD}}^1(\mathcal{C}_{T^n}; \mathbb{Z})$ is given by

$$\text{pr}_1^*(e_i) - \text{pr}_2^*(e_i) \quad (1 \leq i \leq n).$$

The examples above exhibit three regimes: discrete code spaces give only degree-zero classes, spheres and bouquets give two-step patterns, and tori give simultaneous nonvanishing in many degrees. In the torus case, one can think of the rank formula as counting those exterior monomials on $T^n \times T^n$ that are genuinely “bipartite” and therefore do not come from pulling back a class along the diagonal. In every case the decoded fibers are contractible, so the nontrivial classes come entirely from how the diagonal Δ_B sits inside $B \times B$. This is why defect cohomology sees global code-space geometry rather than merely the homotopy type of each decoded fiber, and it is exactly this feature that later drives the thickening and non-reducibility arguments.

5 Thickening and realization in Tarski universes

The constant families \mathcal{C}_B are typically not themselves Tarski universes. To import the same classes into structured universes, start with a Tarski universe

$$\mathcal{U} = (U, \text{El})$$

and define its *thickening* by a type B to be

$$\mathcal{U} \boxtimes B := (U \times B, \text{El} \circ \text{pr}_1).$$

Intuitively, thickening appends an auxiliary label $b : B$ to each code without changing what the code decodes to. Equality of codes now requires both the old code and the new label to match, whereas equivalence of decoded types still depends only on the old code. Thus thickening leaves the semantic fibers unchanged while enlarging the room in which univalence can fail. This is exactly the kind of controlled deformation needed to transport explicit defect classes into structured universes.

Two observations make the construction useful. First, if B is pointed, the closure operations of a structured universe can be transported by applying the old constructor in the first coordinate and resetting the second coordinate to the basepoint. Second, for arbitrary B , the chosen code for 1 embeds the constant family \mathcal{C}_B into the thickening, and this produces a retract on defect spaces.

Proposition 5.1. *Let Σ be an algebraic universe signature containing a code for 1. If \mathcal{U} is a Σ -universe and B is pointed, then $\mathcal{U} \boxtimes B$ admits a canonical structure of Σ -universe.*

Theorem 5.2. *Let $\mathcal{U} = (U, \text{El})$ be a Tarski universe with a chosen code $\hat{1} : U$ satisfying $\text{El}(\hat{1}) \equiv 1$, and let B be any type. Then $D(\mathcal{C}_B)$ is a retract of $D(\mathcal{U} \boxtimes B)$. Consequently, for every abelian group A and every $n \geq 0$,*

$$H_{\text{UD}}^n(\mathcal{C}_B; A)$$

is a direct summand of

$$H_{\text{UD}}^n(\mathcal{U} \boxtimes B; A).$$

The retraction has a simple interpretation. Freezing the original universe coordinate at the chosen code $\hat{1}$ embeds the constant family on B into the thickening, while forgetting that coordinate projects back. Hence any defect class created purely by the added B -labels survives inside the larger structured universe rather than being absorbed by its pre-existing codes.

Corollary 5.3. *Let $\mathcal{U} = (U, \text{El})$ be a Tarski universe containing a chosen code for 1. Then, for every $m \geq 1$,*

$$H_{\text{UD}}^m(\mathcal{U} \boxtimes S^m; \mathbb{Z}) \neq 0,$$

and for $m = 0$,

$$H_{\text{UD}}^0(\mathcal{U} \boxtimes S^0; \mathbb{Z}) \neq 0.$$

Corollary 5.4. *Let $\mathcal{U} = (U, \text{El})$ be a Tarski universe containing a chosen code for 1.*

(i) *For every discrete type F_n with $n \geq 1$ points,*

$$\mathbb{Z}^{n(n-1)}$$

occurs as a direct summand of

$$H_{\text{UD}}^0(\mathcal{U} \boxtimes F_n; \mathbb{Z}).$$

(ii) *For every $r \geq 1$ and $m \geq 1$,*

$$\mathbb{Z}^r$$

and

$$\mathbb{Z}^{r^2}$$

occur as direct summands of

$$H_{\text{UD}}^m(\mathcal{U} \boxtimes W_{r,m}; \mathbb{Z}) \quad \text{and} \quad H_{\text{UD}}^{2m}(\mathcal{U} \boxtimes W_{r,m}; \mathbb{Z}),$$

respectively.

(iii) *For every $n \geq 1$ and every $1 \leq k \leq 2n$, a free abelian group of rank*

$$\binom{2n}{k} - \binom{n}{k}$$

occurs as a direct summand of

$$H_{\text{UD}}^k(\mathcal{U} \boxtimes T^n; \mathbb{Z}).$$

In particular,

$$H_{\text{UD}}^k(\mathcal{U} \boxtimes T^n; \mathbb{Z}) \neq 0 \quad (1 \leq k \leq 2n).$$

Proof. Combine Theorem 5.2 with Proposition 4.4, Theorem 4.5, and Theorem 4.6. \square

The sphere case is the minimal realization needed later, while the discrete, bouquet, and torus cases show that the same mechanism imports much richer patterns. The proof constructs maps

$$\alpha_B : D(\mathcal{C}_B) \rightarrow D(\mathcal{U} \boxtimes B), \quad \beta_B : D(\mathcal{U} \boxtimes B) \rightarrow D(\mathcal{C}_B)$$

with $\beta_B \circ \alpha_B \simeq \text{id}$, so every constant-family class survives inside the thickening. In this sense, thickening is a minimal deformation: it leaves decoded types unchanged while modifying only the code-space geometry that univalence constrains.

6 Structured-model comparison

Fix an algebraic universe signature Σ containing a code for 1. By a *structured model* of HoTT_Σ we mean an ambient semantics together with a chosen interpretation

$$\mathcal{U}_0 = (U_0, \text{El}_0)$$

of the distinguished universe symbol. The theory $\text{HoTT}_\Sigma + \text{UA}$ additionally requires that this chosen universe be univalent. The defect invariant is therefore attached to the distinguished universe. This is the semantic payoff of the previous sections: the ambient homotopy theory may stay fixed while the chosen universe changes, and defect cohomology responds to that change.

Theorem 6.1. *Fix an algebraic signature Σ containing a code for 1.*

(i) *For every structured model*

$$(\mathcal{M}, \mathcal{U}_0) \models \text{HoTT}_\Sigma + \text{UA},$$

every abelian group A , every group G , and every $n \geq 0$, one has

$$H_{\text{UD}}^n(\mathcal{U}_0; A) = 0 \quad \text{and} \quad H_{\text{UD,na}}^1(\mathcal{U}_0; G) = *.$$

(ii) *Assume that the ambient semantics supports spheres S^m for all $m \geq 0$. Then, for every structured model*

$$(\mathcal{M}, \mathcal{U}_0) \models \text{HoTT}_\Sigma$$

and every $m \geq 1$, the thickened universe $\mathcal{U}_0 \boxtimes S^m$ is again a Σ -universe and satisfies

$$H_{\text{UD}}^m(\mathcal{U}_0 \boxtimes S^m; \mathbb{Z}) \neq 0.$$

For $m = 0$,

$$H_{\text{UD}}^0(\mathcal{U}_0 \boxtimes S^0; \mathbb{Z}) \neq 0.$$

(iii) *Consequently, the distinguished-universe invariant*

$$\mathcal{U} \longmapsto H_{\text{UD}}^*(\mathcal{U}; \mathbb{Z})$$

vanishes on every structured model of $\text{HoTT}_\Sigma + \text{UA}$ and is nontrivial on the thickened universes from (ii). In particular, it distinguishes each such thickening from any univalent distinguished universe.

In particular, thickening does not merely manufacture examples in new ambient models. It produces new distinguished universes inside the same model, with the same surrounding semantics but a different geometry of code-space versus equivalence-space comparison.

Corollary 6.2. *Let*

$$(\mathcal{M}, \mathcal{U}_0) \models \text{HoTT}_\Sigma + \text{UA}.$$

Then for every $m \geq 0$, the same ambient model \mathcal{M} supports two structured HoTT_Σ -models,

$$(\mathcal{M}, \mathcal{U}_0) \quad \text{and} \quad (\mathcal{M}, \mathcal{U}_0 \boxtimes S^m),$$

whose distinguished-universe defect invariants are not isomorphic.

Theorem 6.1 is stated with spheres because spheres are the minimal pointed types needed to force one prescribed degree of nonvanishing. When the ambient semantics also supports pointed finite discrete types, bouquets $W_{r,m}$, or tori T^n , the same argument combined with Corollary 5.4 yields structured models whose distinguished-universe defect cohomology exhibits the corresponding degree-zero, two-step, or full-range rank patterns. In that sense the additional examples from Section 4 are not side calculations: they describe the richer separation profiles available once the semantics contains the requisite pointed code spaces.

7 Non-reducibility to fiberwise cohomology

This section shows that H_{UD}^* is not a fiberwise invariant. Ordinary cohomology of the decoded fibers remembers only the individual homotopy types $\text{El}(u)$; it does not remember how the code space U sits inside the equivalence space $\text{Eq}(\mathcal{U})$. The defect cofiber retains that global arrangement, and the following comparison already separates two families with identical contractible fibers. In short, defect cohomology is an invariant of presentation, not merely of the multiset of fiber homotopy types.

Theorem 7.1. *The graded invariant $H_{\text{UD}}^*(-; \mathbb{Z})$ is not determined by the homotopy types, or by the ordinary cohomology groups, of the decoded fibers. Concretely,*

$$\mathcal{C}_1 = (1, \underline{1}) \quad \text{and} \quad \mathcal{C}_{S^1} = (S^1, \underline{1})$$

have identical contractible decoded fibers, but

$$H_{\text{UD}}^*(\mathcal{C}_1; \mathbb{Z}) = 0, \quad H_{\text{UD}}^1(\mathcal{C}_{S^1}; \mathbb{Z}) \cong \mathbb{Z}.$$

The example is intentionally extreme: both families decode every code to the same contractible type, so fiberwise invariants see no difference at all. The entire distinction comes from the geometry of the code space and from how its diagonal sits inside the corresponding equivalence space.

Proposition 7.2. *The families*

$$\mathcal{C}_1 = (1, \underline{1}) \quad \text{and} \quad \mathcal{C}_{S^1} = (S^1, \underline{1})$$

are fiberwise constant at the same type 1, but their defect spaces are not equivalent. More precisely,

$$D(\mathcal{C}_1) \simeq 1, \quad D(\mathcal{C}_{S^1}) \simeq \text{Cofib}(\Delta_{S^1}),$$

and

$$H_{\text{UD}}^1(\mathcal{C}_{S^1}; \mathbb{Z}) \cong \mathbb{Z}.$$

This example clarifies the essential point: even when the decoded fibers carry no cohomological information at all, the defect space can remain highly nontrivial. H_{UD}^* is sensitive to univalence and invisible from the fibers alone.

8 Conclusion

The defect-space construction turns the comparison map attached to a chosen type family into a canonical pointed type. Synthetic cohomology of that type then yields invariants that vanish for univalent families, can be computed explicitly for simple non-univalent families, and can be transported into structured Tarski universes by thickening. The examples show that these invariants detect genuinely global geometry of universe presentations rather than merely fiberwise cohomology. In this sense, defect cohomology provides a bridge between the design of universe presentations and ordinary homotopy-theoretic computation: it turns the abstract comparison underlying univalence into an explicit space whose classes can be calculated and compared.

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A Basic proofs

This appendix collects the omitted proofs and a few supporting technical statements that are used there.

Lemma A.1. *The map*

$$\tau_U : U \rightarrow P(U), \quad \tau_U(u) := (u, u, \text{refl}_u),$$

is an equivalence.

Proof. Define

$$\pi_1 : P(U) \rightarrow U, \quad \pi_1(u, v, p) := u.$$

Then

$$(\pi_1 \circ \tau_U)(u) = u$$

judgmentally, so $\pi_1 \circ \tau_U = \text{id}_U$.

It remains to construct a homotopy

$$\tau_U \circ \pi_1 \sim \text{id}_{P(U)}.$$

Let $x = (u, v, p) : P(U)$. We must produce a path

$$\tau_U(\pi_1(x)) = (u, u, \text{refl}_u) = (u, v, p)$$

in $P(U)$. We define such a path by induction on p . In the reflexivity case $p \equiv \text{refl}_u$, the required path is simply

$$\text{refl}_{(u, u, \text{refl}_u)}.$$

By path induction, this yields the desired homotopy for all $x : P(U)$. Hence τ_U is an equivalence with quasi-inverse π_1 . \square

Proof of Proposition 2.3. By definition, the family \mathcal{U} is univalent exactly when the map

$$\text{idtoeq}_{\mathcal{U}} : P(U) \rightarrow \text{Eq}(\mathcal{U})$$

is an equivalence. By Lemma A.1, the map

$$\tau_U : U \rightarrow P(U)$$

is an equivalence. Therefore, by the 2-out-of-3 property for equivalences,

$$\text{idtoeq}_{\mathcal{U}} \text{ is an equivalence} \iff \text{idtoeq}_{\mathcal{U}} \circ \tau_U = \iota_{\mathcal{U}} \text{ is an equivalence.}$$

This proves the first statement. If \mathcal{U} is univalent, then $\iota_{\mathcal{U}}$ is an equivalence, and the cofiber of an equivalence is contractible. Hence $D(\mathcal{U})$ is contractible. \square

Proof of Proposition 2.6. By definition of relative cohomology,

$$H^n(\mathrm{Eq}(\mathcal{Q}), U; A) = \widetilde{H}^n(\mathrm{Cofib}(\iota_{\mathcal{Q}}); A).$$

By Definition 2.2,

$$\mathrm{Cofib}(\iota_{\mathcal{Q}}) = D(\mathcal{Q}).$$

Substituting yields the claim. \square

Proof of Proposition 2.8. For each $u : U$, we have

$$\mathrm{Eq}(F, \phi)(\iota_{\mathcal{Q}}(u)) = (F(u), F(u), \phi_u \circ \mathrm{id}_{\mathrm{El}(u)} \circ \phi_u^{-1}).$$

There is a canonical path in $\mathrm{Eq}(\mathcal{V})$ from the right-hand side to

$$(F(u), F(u), \mathrm{id}_{\mathrm{El}'(F(u))}) = \iota_{\mathcal{V}}(F(u)).$$

Hence the square

$$\begin{array}{ccc} U & \xrightarrow{\iota_{\mathcal{Q}}} & \mathrm{Eq}(\mathcal{Q}) \\ \downarrow F & & \downarrow \mathrm{Eq}(F, \phi) \\ V & \xrightarrow{\iota_{\mathcal{V}}} & \mathrm{Eq}(\mathcal{V}) \end{array}$$

commutes up to homotopy. By functoriality of homotopy cofibers with respect to homotopy-commuting squares, we obtain a canonical pointed map

$$D(F, \phi) : D(\mathcal{Q}) \rightarrow D(\mathcal{V}).$$

Compatibility with identities and composition is inherited from the corresponding compatibility of the induced squares and the universal property of the cofiber. Applying reduced cohomology and the pointed homotopy-class functor $[-, BG]_*$ yields the stated maps on H_{UD}^n and $H_{\mathrm{UD}, \mathrm{na}}^1$. \square

Proof of Proposition 2.11. For each $u, v : U$, let

$$c_{u,v} : (\mathrm{El}(u) \simeq \mathrm{El}(v)) \rightarrow (\mathrm{El}'(F(u)) \simeq \mathrm{El}'(F(v)))$$

be the conjugation map

$$c_{u,v}(e) := \phi_v \circ e \circ \phi_u^{-1}.$$

Its inverse is

$$c_{u,v}^{-1}(e') := \phi_v^{-1} \circ e' \circ \phi_u,$$

so each $c_{u,v}$ is an equivalence. Therefore the induced map

$$\sum_{u,v:U} (\mathrm{El}(u) \simeq \mathrm{El}(v)) \longrightarrow \sum_{u,v:U} (\mathrm{El}'(F(u)) \simeq \mathrm{El}'(F(v)))$$

is an equivalence. Since F is an equivalence, so is

$$F \times F : U \times U \rightarrow V \times V.$$

Reindexing the family

$$(v_1, v_2) \longmapsto (\mathrm{El}'(v_1) \simeq \mathrm{El}'(v_2))$$

along the equivalence $F \times F$ induces an equivalence

$$\sum_{u,v:U} (\mathrm{El}'(F(u)) \simeq \mathrm{El}'(F(v))) \longrightarrow \sum_{v_1, v_2:V} (\mathrm{El}'(v_1) \simeq \mathrm{El}'(v_2)).$$

The composite of these two equivalences is exactly $\mathrm{Eq}(F, \phi)$. Hence $\mathrm{Eq}(F, \phi)$ is an equivalence. \square

Proof of Corollary 2.12. By Definition 2.9, the map

$$F : U \rightarrow V$$

is an equivalence, and by Proposition 2.11, so is

$$\mathrm{Eq}(F, \phi) : \mathrm{Eq}(\mathcal{U}) \rightarrow \mathrm{Eq}(\mathcal{V}).$$

The square

$$\begin{array}{ccc} U & \xrightarrow{\iota_{\mathcal{U}}} & \mathrm{Eq}(\mathcal{U}) \\ \downarrow F & & \downarrow \mathrm{Eq}(F, \phi) \\ V & \xrightarrow{\iota_{\mathcal{V}}} & \mathrm{Eq}(\mathcal{V}) \end{array}$$

commutes up to homotopy by the calculation already used in the proof of Proposition 2.8. Since cofibers are homotopy pushouts and homotopy pushouts are invariant under equivalences of diagrams, the induced pointed map

$$D(F, \phi) : D(\mathcal{U}) \rightarrow D(\mathcal{V})$$

is an equivalence. The claim about H_{UD}^n follows from homotopy invariance of reduced cohomology, and the claim about $H_{\mathrm{UD,na}}^1$ follows from the same argument applied to the functor $[-, BG]_*$. \square

Proof of Theorem 2.13. By Proposition 2.3, univalence of \mathcal{U} implies that $D(\mathcal{U})$ is contractible. Reduced cohomology of a contractible pointed type vanishes in every degree, so

$$H_{\mathrm{UD}}^n(\mathcal{U}; A) = 0$$

for all abelian groups A and all $n \geq 0$. Likewise, every pointed map $D(\mathcal{U}) \rightarrow BG$ is pointed-homotopic to the constant map, so

$$H_{\mathrm{UD,na}}^1(\mathcal{U}; G) = *$$

for every group G . \square

Proof of Proposition 2.14. Part (i) follows from Proposition 2.8 by applying the pointed homotopy-class functor $[-, BG]_*$.

For part (ii), if G is abelian, then BG is a pointed Eilenberg–MacLane type $K(G, 1)$. Hence

$$[D(\mathcal{U}), BG]_* = [D(\mathcal{U}), K(G, 1)]_* = \tilde{H}^1(D(\mathcal{U}); G) = H_{\mathrm{UD}}^1(\mathcal{U}; G).$$

For part (iii), if \mathcal{U} is univalent, then Theorem 2.13 already gives

$$H_{\mathrm{UD,na}}^1(\mathcal{U}; G) = *.$$

\square

B Detection on truncated defect spaces

Proof of Corollary 3.2. If one of the displayed invariants is nontrivial, then $D(\mathcal{U})$ is not contractible by Theorem 3.1. Since univalence implies contractibility of $D(\mathcal{U})$ by Proposition 2.3, the type family \mathcal{U} cannot be univalent. \square

Proof of Theorem 3.1. The implication (i) \Rightarrow (ii) is immediate. If $D(\mathcal{U})$ is contractible, then all of its reduced cohomology groups vanish, and every pointed mapping set out of $D(\mathcal{U})$ is trivial. Thus

$$H_{\text{UD}}^0(\mathcal{U}; \mathbb{Z}) = 0, \quad H_{\text{UD,na}}^1(\mathcal{U}; G) = *, \quad H_{\text{UD}}^n(\mathcal{U}; A) = 0$$

for all G , all abelian A , and all $2 \leq n \leq N$.

It remains to prove (ii) \Rightarrow (i). We argue by contraposition. Assume that $D := D(\mathcal{U})$ is not contractible.

We first consider the case where D is not connected. By the standard description of reduced degree-zero cohomology,

$$\tilde{H}^0(D; \mathbb{Z}) = 0$$

if and only if D is connected. Hence nonconnectedness implies

$$H_{\text{UD}}^0(\mathcal{U}; \mathbb{Z}) = \tilde{H}^0(D; \mathbb{Z}) \neq 0,$$

so condition (ii)(a) fails.

We may therefore assume that D is connected. If $N = 0$, then a connected pointed 0-type is contractible, contradicting our assumption. Hence $N \geq 1$.

Consider the Postnikov tower

$$\tau_{\leq 0}D \longleftarrow \tau_{\leq 1}D \longleftarrow \cdots \longleftarrow \tau_{\leq N}D = D.$$

Since D is connected, $\tau_{\leq 0}D$ is contractible. Since $D = \tau_{\leq N}D$ is not contractible, there exists a least integer n with $1 \leq n \leq N$ such that $\tau_{\leq n}D$ is not contractible. By minimality, $\tau_{\leq n-1}D$ is contractible.

By the standard Postnikov fiber sequence, the map

$$\tau_{\leq n}D \longrightarrow \tau_{\leq n-1}D$$

has fiber $K(\pi_n D, n)$. Since the codomain is contractible, it follows that

$$\tau_{\leq n}D \simeq K(\pi_n D, n),$$

and in particular $\pi_n D \neq 0$.

If $n = 1$, write

$$G := \pi_1 D.$$

Then

$$\tau_{\leq 1}D \simeq BG.$$

The truncation map

$$q : D \rightarrow \tau_{\leq 1}D \simeq BG$$

defines a class

$$[q] \in [D, BG]_* = H_{\text{UD,na}}^1(\mathcal{U}; G).$$

This class is nontrivial: the map q induces an isomorphism on π_1 , whereas the constant pointed map induces the trivial homomorphism on π_1 . Hence

$$H_{\text{UD,na}}^1(\mathcal{U}; G) \neq *,$$

so condition (ii)(b) fails.

If $n \geq 2$, write

$$A := \pi_n D.$$

Then

$$\tau_{\leq n} D \simeq K(A, n).$$

The truncation map

$$q : D \rightarrow \tau_{\leq n} D \simeq K(A, n)$$

defines a class

$$[q] \in [D, K(A, n)]_* = \tilde{H}^n(D; A) = H_{\text{UD}}^n(\mathcal{U}; A).$$

Again this class is nontrivial, because q induces an isomorphism on π_n , whereas a nullhomotopic map induces the zero map on π_n . Thus

$$H_{\text{UD}}^n(\mathcal{U}; A) \neq 0,$$

so condition (ii)(c) fails.

In all cases, failure of contractibility of $D(\mathcal{U})$ forces failure of condition (ii). This proves (ii) \Rightarrow (i). \square

C Constant unit families

Proof of Proposition 4.1. By definition,

$$\text{Eq}(\mathcal{C}_B) = \sum_{b, b': B} (1 \simeq 1).$$

Since the type of self-equivalences of 1 is contractible, projection onto the first two coordinates induces an equivalence

$$\text{Eq}(\mathcal{C}_B) \simeq B \times B.$$

Under this equivalence, the point

$$\iota_{\mathcal{C}_B}(b) = (b, b, \text{id}_1)$$

corresponds exactly to (b, b) . Hence $\iota_{\mathcal{C}_B}$ identifies with the diagonal map Δ_B . Therefore

$$D(\mathcal{C}_B) \simeq \text{Cofib}(\Delta_B).$$

Finally, Proposition 2.6 identifies

$$H_{\text{UD}}^n(\mathcal{C}_B; A)$$

with the relative cohomology of $\Delta_B : B \rightarrow B \times B$, namely

$$H^n(B \times B, B; A).$$

\square

Proof of Theorem 4.2. Set

$$X := S^m \times S^m, \quad A := S^m,$$

with $A \rightarrow X$ the diagonal map Δ_{S^m} . By Proposition 4.1,

$$H_{\text{UD}}^k(\mathcal{C}_{S^m}; \mathbb{Z}) \cong H^k(X, A; \mathbb{Z}).$$

We use the long exact sequence of the pair (X, A) :

$$\cdots \rightarrow H^{k-1}(A; \mathbb{Z}) \rightarrow H^k(X, A; \mathbb{Z}) \rightarrow H^k(X; \mathbb{Z}) \xrightarrow{\Delta^*} H^k(A; \mathbb{Z}) \rightarrow H^{k+1}(X, A; \mathbb{Z}) \rightarrow \cdots$$

The integral cohomology of $A = S^m$ is

$$H^r(A; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & r = 0, m, \\ 0, & \text{otherwise.} \end{cases}$$

By the Künneth theorem,

$$H^r(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & r = 0, 2m, \\ \mathbb{Z} \oplus \mathbb{Z}, & r = m, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$u \in H^m(S^m; \mathbb{Z})$$

be a generator. Then

$$H^m(X; \mathbb{Z}) = \mathbb{Z} \text{pr}_1^*(u) \oplus \mathbb{Z} \text{pr}_2^*(u),$$

and the diagonal pullback satisfies

$$\Delta^*(\text{pr}_1^*(u)) = u = \Delta^*(\text{pr}_2^*(u)).$$

Hence

$$\Delta^* : H^m(X; \mathbb{Z}) \rightarrow H^m(A; \mathbb{Z})$$

is the addition map

$$\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}, \quad (a, b) \mapsto a + b.$$

We first treat the case $m \geq 2$. Since X and A are connected,

$$H^0(X; \mathbb{Z}) \xrightarrow{\cong} H^0(A; \mathbb{Z}),$$

so

$$H^0(X, A; \mathbb{Z}) = 0 \quad \text{and} \quad H^1(X, A; \mathbb{Z}) = 0.$$

For each k with $1 < k < m$, the neighboring groups in the long exact sequence vanish, so

$$H^k(X, A; \mathbb{Z}) = 0.$$

In degree m , exactness gives

$$0 \rightarrow H^m(X, A; \mathbb{Z}) \rightarrow \mathbb{Z}^2 \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z} \rightarrow H^{m+1}(X, A; \mathbb{Z}) \rightarrow 0.$$

Therefore

$$H^m(X, A; \mathbb{Z}) \cong \ker(\mathbb{Z}^2 \rightarrow \mathbb{Z}) \cong \mathbb{Z},$$

generated by $\text{pr}_1^*(u) - \text{pr}_2^*(u)$, and

$$H^{m+1}(X, A; \mathbb{Z}) = 0.$$

For $m + 1 < k < 2m$, all neighboring groups again vanish, hence

$$H^k(X, A; \mathbb{Z}) = 0.$$

Finally, in degree $2m$, the relevant part of the long exact sequence reduces to

$$0 \rightarrow H^{2m}(X, A; \mathbb{Z}) \rightarrow H^{2m}(X; \mathbb{Z}) \rightarrow 0,$$

since $H^{2m-1}(A; \mathbb{Z}) = 0 = H^{2m}(A; \mathbb{Z})$. Thus

$$H^{2m}(X, A; \mathbb{Z}) \cong H^{2m}(X; \mathbb{Z}) \cong \mathbb{Z}.$$

For $k > 2m$, both $H^k(X; \mathbb{Z})$ and $H^{k-1}(A; \mathbb{Z})$ vanish, so

$$H^k(X, A; \mathbb{Z}) = 0.$$

This proves the theorem for $m \geq 2$.

For $m = 1$, we have $X = T^2$ and $A = S^1$. The same argument applies, except that the segment around degree 1 is

$$H^0(X; \mathbb{Z}) \rightarrow H^0(A; \mathbb{Z}) \rightarrow H^1(X, A; \mathbb{Z}) \rightarrow H^1(X; \mathbb{Z}) \xrightarrow{\Delta^*} H^1(A; \mathbb{Z}) \rightarrow H^2(X, A; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z}) \rightarrow 0.$$

The first map is an isomorphism because X and A are connected. Hence

$$H^1(X, A; \mathbb{Z}) \cong \ker(\Delta^*).$$

Since

$$\Delta^* : \mathbb{Z}^2 \rightarrow \mathbb{Z}$$

is again the addition map,

$$H^1(X, A; \mathbb{Z}) \cong \mathbb{Z}.$$

Because Δ^* is surjective, the connecting map

$$H^1(A; \mathbb{Z}) \rightarrow H^2(X, A; \mathbb{Z})$$

is zero, so exactness yields an injection

$$H^2(X, A; \mathbb{Z}) \hookrightarrow H^2(X; \mathbb{Z}) \cong \mathbb{Z}.$$

Since the next group $H^2(A; \mathbb{Z})$ vanishes, this injection is also surjective. Therefore

$$H^2(X, A; \mathbb{Z}) \cong \mathbb{Z}.$$

All remaining groups vanish for degree reasons. This completes the proof. □

Proof of Proposition 4.3. By Proposition 4.1,

$$D(\mathcal{C}_{S^0}) \simeq \text{Cofib}(\Delta_{S^0}),$$

where

$$\Delta_{S^0} : 2 \rightarrow 2 \times 2.$$

The type 2×2 consists of four points:

$$(0,0), (0,1), (1,0), (1,1).$$

Passing to the cofiber identifies the two diagonal points $(0,0)$ and $(1,1)$ with the cone point, while the two off-diagonal points remain distinct. Thus $D(\mathcal{C}_{S^0})$ is a pointed discrete type with three connected components.

The reduced degree-zero cohomology of a pointed discrete type with three components is \mathbb{Z}^2 . Hence

$$H_{\text{UD}}^0(\mathcal{C}_{S^0}; \mathbb{Z}) = \tilde{H}^0(D(\mathcal{C}_{S^0}); \mathbb{Z}) \cong \mathbb{Z}^2.$$

Since $D(\mathcal{C}_{S^0})$ is a 0-type, all higher reduced cohomology groups vanish:

$$H_{\text{UD}}^k(\mathcal{C}_{S^0}; \mathbb{Z}) = 0 \quad (k > 0).$$

□

Proof of Proposition 4.4. By Proposition 4.1,

$$D(\mathcal{C}_{F_n}) \simeq \text{Cofib}(\Delta_{F_n}),$$

where

$$\Delta_{F_n} : F_n \rightarrow F_n \times F_n.$$

Because F_n is discrete with n points, the product $F_n \times F_n$ is discrete with n^2 points. Passing to the cofiber collapses the n diagonal points to the cone point and leaves the $n^2 - n = n(n-1)$ off-diagonal points distinct. Thus $D(\mathcal{C}_{F_n})$ is a pointed discrete type with

$$n(n-1) + 1$$

connected components.

The reduced degree-zero cohomology of a pointed discrete type with $N+1$ connected components is \mathbb{Z}^N . Therefore

$$H_{\text{UD}}^0(\mathcal{C}_{F_n}; \mathbb{Z}) = \tilde{H}^0(D(\mathcal{C}_{F_n}); \mathbb{Z}) \cong \mathbb{Z}^{n(n-1)}.$$

Since $D(\mathcal{C}_{F_n})$ is a 0-type, all higher reduced cohomology groups vanish:

$$H_{\text{UD}}^k(\mathcal{C}_{F_n}; \mathbb{Z}) = 0 \quad (k > 0).$$

□

Proof of Theorem 4.5. Set

$$W := W_{r,m}, \quad X := W \times W, \quad Y := W,$$

with $Y \rightarrow X$ given by the diagonal map. By Proposition 4.1,

$$H_{\text{UD}}^k(\mathcal{C}_W; \mathbb{Z}) \cong H^k(X, Y; \mathbb{Z}).$$

Let

$$u_1, \dots, u_r \in H^m(W; \mathbb{Z})$$

be the basis dual to the r sphere summands. The integral cohomology of W is

$$H^q(W; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}^r, & q = m, \\ 0, & \text{otherwise.} \end{cases}$$

Since these groups are torsion free, the Künneth theorem gives

$$H^q(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}^{2r}, & q = m, \\ \mathbb{Z}^{r^2}, & q = 2m, \\ 0, & \text{otherwise.} \end{cases}$$

More explicitly,

$$H^m(X; \mathbb{Z}) = \bigoplus_{i=1}^r \mathbb{Z} \text{pr}_1^*(u_i) \oplus \bigoplus_{i=1}^r \mathbb{Z} \text{pr}_2^*(u_i),$$

and

$$H^{2m}(X; \mathbb{Z}) = \bigoplus_{1 \leq i, j \leq r} \mathbb{Z} \text{pr}_1^*(u_i) \smile \text{pr}_2^*(u_j).$$

The diagonal pullback satisfies

$$\Delta^*(\text{pr}_1^*(u_i)) = u_i = \Delta^*(\text{pr}_2^*(u_i)),$$

so under the identification

$$H^m(X; \mathbb{Z}) \cong \mathbb{Z}^r \oplus \mathbb{Z}^r, \quad H^m(Y; \mathbb{Z}) \cong \mathbb{Z}^r,$$

the map

$$\Delta^* : H^m(X; \mathbb{Z}) \rightarrow H^m(Y; \mathbb{Z})$$

is componentwise addition:

$$(a, b) \mapsto a + b.$$

We first treat the case $m \geq 2$. Since X and Y are connected,

$$H^0(X; \mathbb{Z}) \xrightarrow{\cong} H^0(Y; \mathbb{Z}),$$

hence

$$H^0(X, Y; \mathbb{Z}) = 0 \quad \text{and} \quad H^1(X, Y; \mathbb{Z}) = 0.$$

For each k with $1 < k < m$, all neighboring groups in the long exact sequence of the pair (X, Y) vanish, so

$$H^k(X, Y; \mathbb{Z}) = 0.$$

In degree m , exactness gives

$$0 \rightarrow H^m(X, Y; \mathbb{Z}) \rightarrow \mathbb{Z}^{2r} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}^r \rightarrow H^{m+1}(X, Y; \mathbb{Z}) \rightarrow 0.$$

Therefore

$$H^m(X, Y; \mathbb{Z}) \cong \mathbb{Z}^r,$$

with basis represented by

$$\text{pr}_1^*(u_i) - \text{pr}_2^*(u_i) \quad (1 \leq i \leq r),$$

and

$$H^{m+1}(X, Y; \mathbb{Z}) = 0.$$

For $m+1 < k < 2m$, all neighboring groups again vanish, hence

$$H^k(X, Y; \mathbb{Z}) = 0.$$

In degree $2m$, the relevant part of the long exact sequence is

$$0 \rightarrow H^{2m}(X, Y; \mathbb{Z}) \rightarrow H^{2m}(X; \mathbb{Z}) \rightarrow 0,$$

because $H^{2m-1}(Y; \mathbb{Z}) = 0 = H^{2m}(Y; \mathbb{Z})$. Therefore

$$H^{2m}(X, Y; \mathbb{Z}) \cong \mathbb{Z}^{r^2},$$

with basis represented by the classes

$$\text{pr}_1^*(u_i) \smile \text{pr}_2^*(u_j) \quad (1 \leq i, j \leq r).$$

Finally, $H^k(X, Y; \mathbb{Z}) = 0$ for $k > 2m$ for degree reasons.

This proves the theorem for $m \geq 2$.

For $m = 1$, the same calculation applies except in low degree. The segment of the long exact sequence around degree 1 is

$$H^0(X; \mathbb{Z}) \rightarrow H^0(Y; \mathbb{Z}) \rightarrow H^1(X, Y; \mathbb{Z}) \rightarrow H^1(X; \mathbb{Z}) \xrightarrow{\Delta^*} H^1(Y; \mathbb{Z}) \rightarrow H^2(X, Y; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z}) \rightarrow 0.$$

The first map is an isomorphism because X and Y are connected, so

$$H^1(X, Y; \mathbb{Z}) \cong \ker(\Delta^*) \cong \mathbb{Z}^r,$$

again generated by the classes

$$\text{pr}_1^*(u_i) - \text{pr}_2^*(u_i).$$

Since Δ^* is surjective, the connecting map

$$H^1(Y; \mathbb{Z}) \rightarrow H^2(X, Y; \mathbb{Z})$$

is zero. Exactness then yields an isomorphism

$$H^2(X, Y; \mathbb{Z}) \cong H^2(X; \mathbb{Z}) \cong \mathbb{Z}^{r^2},$$

with basis

$$\text{pr}_1^*(u_i) \smile \text{pr}_2^*(u_j) \quad (1 \leq i, j \leq r).$$

All remaining groups vanish for degree reasons. This completes the proof. \square

Proof of Theorem 4.6. Set

$$X := T^n \times T^n, \quad Y := T^n,$$

with $Y \rightarrow X$ the diagonal inclusion. By Proposition 4.1,

$$H_{\text{UD}}^k(\mathcal{C}_{T^n}; \mathbb{Z}) \cong H^k(X, Y; \mathbb{Z}).$$

Let

$$e_1, \dots, e_n \in H^1(T^n; \mathbb{Z})$$

be the standard basis. Iterating the Künneth theorem gives the cohomology ring description

$$H^*(Y; \mathbb{Z}) \cong \Lambda(e_1, \dots, e_n),$$

so, in particular,

$$\text{rk} H^k(Y; \mathbb{Z}) = \binom{n}{k}.$$

Write

$$x_i := \text{pr}_1^*(e_i), \quad y_i := \text{pr}_2^*(e_i) \quad (1 \leq i \leq n).$$

Then

$$H^*(X; \mathbb{Z}) \cong \Lambda(x_1, \dots, x_n, y_1, \dots, y_n),$$

and therefore

$$\text{rk} H^k(X; \mathbb{Z}) = \binom{2n}{k}.$$

The diagonal pullback satisfies

$$\Delta^*(x_i) = e_i = \Delta^*(y_i),$$

and because

$$\text{pr}_1 \circ \Delta = \text{id}_Y,$$

the map

$$\Delta^* : H^k(X; \mathbb{Z}) \rightarrow H^k(Y; \mathbb{Z})$$

is split surjective in every degree, with section pr_1^* . The long exact sequence of the pair (X, Y) therefore breaks into short exact sequences

$$0 \rightarrow H^k(X, Y; \mathbb{Z}) \rightarrow H^k(X; \mathbb{Z}) \xrightarrow{\Delta^*} H^k(Y; \mathbb{Z}) \rightarrow 0.$$

Hence

$$H^k(X, Y; \mathbb{Z}) \cong \ker(\Delta^*)$$

for every k . Since the source and target are free abelian, $H^k(X, Y; \mathbb{Z})$ is free abelian of rank

$$\binom{2n}{k} - \binom{n}{k}.$$

For $k > 2n$, one has $H^k(X; \mathbb{Z}) = 0$, so

$$H^k(X, Y; \mathbb{Z}) = 0.$$

In degree 1, the kernel of

$$\Delta^* : H^1(X; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z})$$

is generated by the n classes

$$x_i - y_i = \text{pr}_1^*(e_i) - \text{pr}_2^*(e_i) \quad (1 \leq i \leq n).$$

Transporting the calculation back through Proposition 4.1 gives the theorem. \square

D Thickening and realization

Proof of Proposition 5.1. For each code constructor of \mathcal{U} , define the corresponding constructor on $U \times B$ by applying the original constructor to the first coordinates and tagging the result with the fixed basepoint $b_0 : B$.

Concretely, if $\widehat{1} : U$ is a chosen code for 1, define

$$\widehat{1}_{\boxtimes} := (\widehat{1}, b_0).$$

If $\widehat{\Pi}$ is a code former for Π -types, define

$$\widehat{\Pi}_{\boxtimes}((u, b), g) := (\widehat{\Pi}(u, \text{pr}_1 \circ g), b_0),$$

where

$$g : \text{El}(u) \rightarrow U \times B.$$

Similarly, if $\widehat{\Sigma}$ is a code former for Σ -types, define

$$\widehat{\Sigma}_{\boxtimes}((u, b), g) := (\widehat{\Sigma}(u, \text{pr}_1 \circ g), b_0).$$

If $\widehat{\text{Id}}$ is a code former for identity types, define

$$\widehat{\text{Id}}_{\boxtimes}((u, b), x, y) := (\widehat{\text{Id}}(u, x, y), b_0).$$

The same recipe applies to every operation symbol in the signature Σ : the first coordinate is computed exactly as in \mathcal{U} and the second coordinate is reset to b_0 .

Since the decoder of the thickened universe ignores the second coordinate, every decoding equation for $\mathcal{U} \boxtimes B$ reduces immediately to the corresponding decoding equation for \mathcal{U} . For example,

$$\text{El}_{\boxtimes}(\widehat{1}_{\boxtimes}) = \text{El}(\widehat{1}) \equiv 1,$$

and

$$\text{El}_{\boxtimes}(\widehat{\Pi}_{\boxtimes}((u, b), g)) = \text{El}(\widehat{\Pi}(u, \text{pr}_1 \circ g)) \equiv \prod_{x : \text{El}(u)} \text{El}((\text{pr}_1 \circ g)(x)) = \prod_{x : \text{El}_{\boxtimes}(u, b)} \text{El}_{\boxtimes}(g(x)).$$

The verification for Σ -types, identity types, and any other constructors in Σ is identical. Thus $\mathcal{U} \boxtimes B$ is again a Σ -universe. \square

Proof of Theorem 5.2. By Proposition 4.1,

$$D(\mathcal{C}_B) \simeq \text{Cofib}(\Delta_B).$$

We construct two homotopy-commuting squares over the maps

$$\Delta_B : B \rightarrow B \times B \quad \text{and} \quad \iota_{\mathcal{U} \boxtimes B} : U \times B \rightarrow \text{Eq}(\mathcal{U} \boxtimes B).$$

First define

$$p : U \times B \rightarrow B, \quad p(u, b) := b.$$

Next define

$$q : \text{Eq}(\mathcal{U} \boxtimes B) \rightarrow B \times B$$

by

$$q((u, b), (v, b'), e) := (b, b').$$

Now define

$$s : B \rightarrow U \times B, \quad s(b) := (\widehat{1}, b),$$

and

$$r : B \times B \rightarrow \text{Eq}(\mathcal{U} \boxtimes B)$$

by

$$r(b, b') := ((\widehat{1}, b), (\widehat{1}, b'), \text{id}_1),$$

where $\text{id}_1 : 1 \simeq 1$ is the identity equivalence.

We claim that the two squares

$$\begin{array}{ccc} B & \xrightarrow{\Delta_B} & B \times B \\ \downarrow s & & \downarrow r \\ U \times B & \xrightarrow{\iota_{\mathcal{U} \boxtimes B}} & \text{Eq}(\mathcal{U} \boxtimes B) \end{array} \quad \begin{array}{ccc} U \times B & \xrightarrow{\iota_{\mathcal{U} \boxtimes B}} & \text{Eq}(\mathcal{U} \boxtimes B) \\ \downarrow p & & \downarrow q \\ B & \xrightarrow{\Delta_B} & B \times B \end{array}$$

commute strictly.

For the left square, we compute

$$r(\Delta_B(b)) = r(b, b) = ((\widehat{1}, b), (\widehat{1}, b), \text{id}_1) = \iota_{\mathcal{U} \boxtimes B}(\widehat{1}, b) = \iota_{\mathcal{U} \boxtimes B}(s(b)).$$

For the right square,

$$q(\iota_{\mathcal{U} \boxtimes B}(u, b)) = q((u, b), (u, b), \text{id}_{E_1(u)}) = (b, b) = \Delta_B(p(u, b)).$$

Moreover,

$$p \circ s = \text{id}_B \quad \text{and} \quad q \circ r = \text{id}_{B \times B}.$$

By functoriality of the homotopy cofiber, the left square induces a pointed map

$$\alpha_B : \text{Cofib}(\Delta_B) \rightarrow \text{Cofib}(\iota_{\mathcal{U} \boxtimes B}),$$

and the right square induces a pointed map

$$\beta_B : \text{Cofib}(\iota_{\mathcal{U} \boxtimes B}) \rightarrow \text{Cofib}(\Delta_B).$$

Using the identifications

$$\text{Cofib}(\Delta_B) \simeq D(\mathcal{C}_B), \quad \text{Cofib}(\iota_{\mathcal{U} \boxtimes B}) = D(\mathcal{U} \boxtimes B),$$

we obtain pointed maps

$$\alpha_B : D(\mathcal{C}_B) \rightarrow D(\mathcal{U} \boxtimes B), \quad \beta_B : D(\mathcal{U} \boxtimes B) \rightarrow D(\mathcal{C}_B).$$

Since the two defining squares compose to the identity square on Δ_B , functoriality of cofibers implies

$$\beta_B \circ \alpha_B \simeq \text{id}_{D(\mathcal{C}_B)}.$$

Thus $D(\mathcal{C}_B)$ is a retract of $D(\mathcal{U} \boxtimes B)$.

Applying reduced cohomology with coefficients in an abelian group A , we obtain

$$(\beta_B \circ \alpha_B)^* = \alpha_B^* \circ \beta_B^* = \text{id}.$$

Hence

$$\beta_B^* : H_{\text{UD}}^n(\mathcal{C}_B; A) \rightarrow H_{\text{UD}}^n(\mathcal{U} \boxtimes B; A)$$

is split injective, so $H_{\text{UD}}^n(\mathcal{C}_B; A)$ is a direct summand of $H_{\text{UD}}^n(\mathcal{U} \boxtimes B; A)$. \square

Proof of Corollary 5.3. For $m \geq 1$, Theorem 4.2 gives

$$H_{\text{UD}}^m(\mathcal{C}_{S^m}; \mathbb{Z}) \cong \mathbb{Z}.$$

By Theorem 5.2, this group embeds as a direct summand of

$$H_{\text{UD}}^m(\mathcal{U} \boxtimes S^m; \mathbb{Z}),$$

so the latter group is nonzero.

For $m = 0$, Proposition 4.3 gives

$$H_{\text{UD}}^0(\mathcal{C}_{S^0}; \mathbb{Z}) \cong \mathbb{Z}^2.$$

Again by Theorem 5.2, this group is a direct summand of

$$H_{\text{UD}}^0(\mathcal{U} \boxtimes S^0; \mathbb{Z}),$$

so that group is nonzero as well. \square

E Separation and non-reducibility

Proof of Theorem 6.1. Part (i) is exactly Theorem 2.13, applied to the distinguished universe \mathcal{U}_0 .

For part (ii), by Proposition 5.1, the thickened universe

$$\mathcal{U}_0 \boxtimes S^m$$

is again a Σ -universe. By Corollary 5.3,

$$H_{\text{UD}}^m(\mathcal{U}_0 \boxtimes S^m; \mathbb{Z}) \neq 0 \quad (m \geq 1),$$

and

$$H_{\text{UD}}^0(\mathcal{U}_0 \boxtimes S^0; \mathbb{Z}) \neq 0.$$

Part (iii) follows immediately. By part (i), every structured model of $\text{HoTT}_\Sigma + \text{UA}$ has vanishing distinguished-universe defect invariant. By part (ii), each thickened universe $\mathcal{U}_0 \boxtimes S^m$ has a nonzero defect class. Hence the invariant distinguishes these explicit thickened universes from any univalent distinguished universe. \square

Proof of Corollary 6.2. By Theorem 6.1(i),

$$H_{\text{UD}}^*(\mathcal{U}_0; \mathbb{Z}) = 0.$$

By Theorem 6.1(ii),

$$H_{\text{UD}}^m(\mathcal{U}_0 \boxtimes S^m; \mathbb{Z}) \neq 0$$

for $m \geq 1$, and

$$H_{\text{UD}}^0(\mathcal{W}_0 \boxtimes S^0; \mathbb{Z}) \neq 0.$$

Therefore the graded invariants of the two distinguished universes are not isomorphic. Since H_{UD}^* is invariant under equivalence of type families by Corollary 2.12, the two distinguished universes are not equivalent. In particular, the two structured models cannot be isomorphic. \square

Proof of Theorem 7.1. We compare two constant unit families:

$$\mathcal{C}_1 = (1, \underline{1}), \quad \mathcal{C}_{S^1} = (S^1, \underline{1}).$$

In both cases, every decoded fiber is the unit type 1. Hence every fiber is contractible, and for every coefficient group A ,

$$\tilde{H}^*(\text{El}(u); A) = 0$$

for every code u in either family.

However, by Proposition 4.1,

$$D(\mathcal{C}_1) \simeq \text{Cofib}(\Delta_1).$$

Since $\Delta_1 : 1 \rightarrow 1 \times 1$ is an equivalence, $\text{Cofib}(\Delta_1)$ is contractible. Therefore

$$H_{\text{UD}}^*(\mathcal{C}_1; \mathbb{Z}) = 0.$$

On the other hand, by Theorem 4.2 with $m = 1$,

$$H_{\text{UD}}^1(\mathcal{C}_{S^1}; \mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad H_{\text{UD}}^2(\mathcal{C}_{S^1}; \mathbb{Z}) \cong \mathbb{Z}.$$

Thus the two families have identical decoded fibers, and hence identical ordinary cohomology of all decoded fibers, but different defect cohomology. This proves the claim. \square

Proof of Proposition 7.2. The first equivalence follows from the fact that

$$\Delta_1 : 1 \rightarrow 1 \times 1$$

is an equivalence, so its cofiber is contractible. The description of $D(\mathcal{C}_{S^1})$ follows from Proposition 4.1. By Theorem 4.2,

$$H_{\text{UD}}^1(\mathcal{C}_{S^1}; \mathbb{Z}) \cong \mathbb{Z} \neq 0.$$

If $D(\mathcal{C}_1) \simeq D(\mathcal{C}_{S^1})$, then by homotopy invariance of reduced cohomology the first group would have to vanish as well, which is impossible. Therefore the two defect spaces are not equivalent. \square