Quantum Coherence Spaces Revisited: A von Neumann (Co)Algebraic Approach

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Introduction and context. In this work we set out to paint a new picture of what a quantum coherence space (QCS) ought to be. See [LZ26] for the full version. We consider finite-dimensional quantum theory and we describe a model of MALL based on the following natural ideas: (1) proofs $P \vdash R$ of formulas with *positive* logical polarities should admit an interpretation as CPTP (completely positive trace-preserving) maps, i.e. as the quantum operations in the *Schrödinger picture* of quantum theory; (2) proofs $M \vdash N$ of formulas with *negative* logical polarities should admit an interpretation as CPU (completely positive unital) maps, i.e. as the quantum operations in the *Heisenberg picture* of quantum theory; (3) the linear logic (LL) duality should coincide with the Heisenberg-Schrödinger duality of quantum theory on polarised formulas. These desiderata naturally lead to the theory of operator spaces [ER00; BL04; Pis03] which has excellent categorical properties [LZ24] and which can be used to construct a model of (full) LL whose duality is compatible with the Heisenberg-Schrödinger duality on the level of objects/formulas [LZ25]. However, in [LZ25], the morphisms/proofs do *not* correspond to the quantum operations in either picture. We address this problem, (which is suggested for future work in [LZ25]) for *finite-dimensional* (f.d.) quantum theory and MALL. It is worth noting that Girard proposed a model of QCSs [Gir04; Gir07a; Gir07b], later studied in [Bar10], however this approach has been criticised, see [Sel04]. One of the main problems being that the notion of morphism between QCSs does not appear to correspond to a *completely positive map* (see also [Bar10, p. 2]).

Background. We work in the setting of finite dimensional (f.d.) *operator spaces*, which are widely seen as the "non-commutative" or "quantised" generalisation of Banach spaces. Given a vector space X, we write $\mathbb{M}_n(X)$ for the vector space consisting of the $n \times n$ matrices with entries in X. A *f.d. operator space* is a f.d. complex vector space X equipped with a sequence of norms $\|-\|_n : \mathbb{M}_n(X) \to [0, \infty)$, one for each $n \in \mathbb{N}$, that satisfy some conditions. A sequence of norms that satisfies these conditions is called an *operator space structure* (o.s.s.) on X. We write $M_n(X)$ for the normed space $(\mathbb{M}_n(X), \| \cdot \|_n)$. Every f.d. operator space X is a Banach space w.r.t. the ground norm $\|-\|_1$ for which we simply write $\|-\|$. We write $M_n = (\mathbb{M}_n(\mathbb{C}), \|-\|_{\mathrm{op}})$ for the space of $n \times n$ complex matrices equipped with the usual operator norm. In fact, M_n is canonically also an operator space with o.s.s. determined via the linear isomorphism $\mathbb{M}_m(M_n) \cong M_{nm}$ and the operator norm on the latter space. A linear map between operator spaces $\varphi : X \to Y$ induces for each $n \in \mathbb{N}$ a linear map $\varphi_n : M_n(X) \to M_n(Y)$ via componentwise application. We say that φ is *completely bounded* (*completely contractive*) if $\|\varphi\|_{\mathrm{cb}} \triangleq \sup \{\|\varphi_n\| \ : \ n \in \mathbb{N}\} < \infty$ ($\|\varphi\|_{\mathrm{cb}} \leq 1$), where $\|\varphi_n\|$ is the standard operator norm of φ_n. See [Pis03; ER00] for introductions to the theory of operator spaces.

Categorical properties of FdOS. We begin our contributions by proving relevant categorical properties of **FdOS**, the category of *f.d. operator spaces* and *linear completely contractive maps*, which constitutes the base upon which we build the successive results. This category has three notable monoidal structures given by: the *projective tensor* $\widehat{\otimes}$, which serves as the multiplicative conjunction; the *injective tensor* $\check{\otimes}$, which serves as the multiplicative disjunction; and the *Haagerup tensor* $\overset{\mathrm{h}}{\otimes}$, which is outside the scope of LL, but very important in operator space theory. We refer to [ER00] for details about the three tensors. The category **FdOS** has a closed symmetric monoidal structure $(\mathbf{FdOS}, \mathbb{C}, \widehat{\otimes}, \mathrm{CB}(-,-))$, where $\mathrm{CB}(-,-)$ is the operator space of completely bounded maps. It also has (co)products given by the direct sum $\overset{\infty}{\oplus}$ ($\overset{1}{\oplus}$), which has underlying vector space given by the direct sum $\oplus$ of vector spaces, and o.s.s. given by the operator space generalizations of the ℓ_∞ and ℓ_1 norms from Banach space theory (see [Pis03] for details). The operator space dual $X^* \triangleq \mathrm{CB}(X, \mathbb{C})$ gives the interpretation of linear negation and it is consistent with the Banach/vector space dual for f.d. operator spaces.

Example 1 *The vector space $\mathbb{M}_n$ can be equipped with another important o.s.s. via the linear isomorphism $\mathbb{M}_n \cong M_n^* :: a \mapsto (b \mapsto \mathrm{tr}(ab))$ and the o.s.s. of the latter space. We write T_n for the resulting operator space. Its ground norm is precisely the trace norm $\|-\|_{\mathrm{tr}}$.*

Theorem 1 *The category* **FdOS** *has finite (co)products, it is $*$-autonomous with $\mathbb{C}$ as global dualizing object and is therefore a model of MALL.*

The Haagerup tensor is outside the scope of Linear Logic. However, it is compatible with the $*$-autonomous structure and it makes **FdOS** into a *BV-category (with negation)* in the sense of [BPS10]. This observation relies on the fact that the Haagerup tensor on f.d. operator spaces, is self-dual, meaning that $X^* \overset{h}{\otimes} Y^* \cong (X \overset{h}{\otimes} Y)^*$, see [Pis03, §5].

von-Neumann (co)algebras. Operator spaces alone do not have sufficient structure for quantum computation. In the Heisenberg picture, we can use *von Neumann algebras (vN-algebras)* to define the relevant quantum operations (i.e. quantum channels). Every such algebra can be equipped with a canonical o.s.s. [ER00], so we may think of them as operator spaces with additional structure. We show that vN-algebras may be equivalently defined as certain kinds of involutive monoid objects in $(\mathbf{FdOS}, \mathbb{C}, \overset{h}{\otimes})$. This novel categorical definition, and the self-duality of the Haagerup tensor, allow us to easily dualise this notion to formulate *von Neumann coalgebras* as certain involutive comonoid objects in $(\mathbf{FdOS}, \mathbb{C}, \overset{h}{\otimes})$, which we use for the Schrödinger picture, allowing for a definition that is independent from vN-algebras.

Proposition 1 *If A is a vN-algebra, then the o.s. dual A^* is a vN-coalgebra in a canonical way, and vice-versa, if C is a vN-coalgebra.*

We can endow the operator space M_n with a vN-algebra structure using the usual matrix multiplication monoid structure and conjugate transpose serving as the involution. We then obtain an isomorphism of vN-coalgebras $T_n \cong M_n^*$ and also a vN-algebra isomorphism $T_n^* \cong M_n$ by duality. It is well-known that all f.d. vN-algebras are of the form $\overset{\infty}{\oplus}_i M_{k_i}$, modulo a vN-algebra isomorphism. We also prove a similar representation result for vN-coalgebras.

Proposition 2 *If C is a vN-coalgebra, then there is a vN-coalgebra isomorphism $C \cong \overset{1}{\oplus}_{0 \leq i \leq n} T_{k_i}$, for some finite sequence of $k_i > 0$, unique up to permutation.*

In order to capture quantum operations in the Heisenberg and Schrödinger pictures, we revisit the notion of *completely positive (CP)* maps for each picture by giving categorical definitions that are Hilbert-space-free, i.e. abstract. We further show that CPU maps are a natural notion of morphism between vN-algebras and that CPTP maps are a natural notion of morphism between vN-coalgebras. We can now define two subcategories of **FdOS**: the subcategory of f.d. vN-algebras and CPU maps, written **H**; and the subcategory of f.d. vN-coalgebras and CPTP maps, written **S**. The category **H** (**S**) is monoidal w.r.t. $\check{\otimes}$ ($\widehat{\otimes}$) and has (co)products given by $\overset{\infty}{\oplus}$ ($\overset{1}{\oplus}$). This enables us to give a categorical formulation of the Heisenberg-Schrödinger duality as follows.

Theorem 2 *We have an equivalence of categories $(\cdot)^* \colon \mathbf{S} \simeq \mathbf{H}^{\mathrm{op}} \colon (\cdot)^*$ where the action on objects of $(\cdot)^*$ is defined as in Proposition 1. Moreover, this equivalence is strong monoidal (and (co)product preserving).*

Revisiting Quantum Coherence Spaces. Now we know that the (sub)categories $\mathbf{S} \hookrightarrow \mathbf{FdOS} \hookleftarrow \mathbf{H}$ have the right categorical structure for a model of MALL whose duality is induced by the Heisenberg-Schrödinger duality. However, neither **S**, nor **H**, is a *full* subcategory of **FdOS**. Luckily, there is a simple solution: a semantic technique of Hyland and Schalk [HS03], based on *gluing and orthogonality*, allows us to carve out a category **Q** from **FdOS**, such that we get fully faithful inclusions $\mathbf{S} \hookrightarrow \mathbf{Q} \hookleftarrow \mathbf{H}$, while preserving all the MALL structure. This is precisely what we do by defining a suitable notion of orthogonality/polarity based on ideas from [HS03].

Definition 1 *Let* **Q** *be the* tight orthogonality category *on* **FdOS** *with* focus $\{id_{\mathbb{C}}\} \subseteq \mathbf{FdOS}(\mathbb{C}, \mathbb{C})$.

For a vN-coalgebra C, we write $P_C = \{\rho \in C \mid \varepsilon(\rho) = 1 \text{ and } \rho \geq 0\}$ for the set of density operators, where ε indicates the counit of C.

Theorem 3 *The category* **Q** *has finite (co)products, it is $*$-autonomous and is therefore a model of MALL. Moreover, the functor $H \colon \mathbf{H} \to \mathbf{Q}$ defined by $H(A) \triangleq (A, \{1_A\})$ and $H(f) \triangleq f$, is fully faithful, strict monoidal w.r.t $\check{\otimes}$, and it strictly preserves finite products. Furthermore, the functor $S \colon \mathbf{S} \to \mathbf{Q}$ defined by $S(C) \triangleq (C, P_C)$ and $S(f) \triangleq f$, is fully faithful, strict monoidal w.r.t $\widehat{\otimes}$, and it strictly preserves finite coproducts.*

Formulas with positive (negative) logical polarities admit natural interpretations as vN-coalgebras $\mathcal{C}, \mathcal{D}$ (vN-algebras $\mathcal{A}, \mathcal{B}$) and proofs between such formulas correspond precisely to the CPTP (CPU) maps. Moreover, this correspondence is preserved by classical and quantum composition in both pictures. See Figure 1 for the interpratations of these.

Q also has interesting properties that are relevant to *pure state* quantum computation, where unitarity is important. Given a unitary matrix $u \in M_n$, we have that the pair $(M_n, \{u\})$ is an object in **Q**. Objects of this form can be composed using the projective tensor in **Q**, as $(M_n, \{u\}) \widehat{\otimes} (M_n, \{v\}) = (M_n \widehat{\otimes} M_n, \{u \otimes v\})$ for unitary matrices $u, v \in M_n$. This enables us to reason about interesting higher-order (pure state) maps in **Q**, such as the pure state *quantum switch*, qsw, which is a complete contraction (see [LZ25, §IV]). Hence, qsw: $(M_n \widehat{\otimes} M_n, \{u \otimes v\}) \to (M_{2n}, \{|0\rangle \langle 0| \otimes (uv) + |1\rangle \langle 1| \otimes (vu)\})$ is a morphism in **Q**. Therefore it may be recognised as a valid (unitary-preserving) higher-order map in **Q**.

Schrödinger Picture	$\mathbf{S} \overset{\text{full}}{\hookrightarrow} \mathbf{Q}$	LL_+
System description	$\mathcal{C}, \mathcal{D}$	P, R
Quantum composition	$\mathcal{C} \widehat{\otimes} \mathcal{D}$	$P \otimes R$
Classical composition	$\mathcal{C} \overset{1}{\oplus} \mathcal{D}$	$P \oplus R$
Quantum operation	$\mathcal{C} \xrightarrow{\text{CPTP}} \mathcal{D}$	$P \vdash R$

Heisenberg Picture	$\mathbf{H} \overset{\text{full}}{\hookrightarrow} \mathbf{Q}$	LL_-
System description	$\mathcal{A}, \mathcal{B}$	N, M
Quantum composition	$\mathcal{A} \check{\otimes} \mathcal{B}$	$N \parr M$
Classical composition	$\mathcal{A} \overset{\infty}{\oplus} \mathcal{B}$	$N \,\&\, M$
Quantum operation	$\mathcal{B} \xrightarrow{\text{CPU}} \mathcal{A}$	$M \vdash N$

Figure 1: Schrödinger/Heisenberg picture and positive/negative logical polarity.

References

[ER00] E.G. Effros and Z.J. Ruan. *Operator Spaces*. London Mathematical Society monographs. Clarendon Press, 2000. ISBN: 9780198534822. URL: https://books.google.fr/books?id=v7mj8Dy84k8C.

[HS03] Martin Hyland and Andrea Schalk. "Glueing and orthogonality for models of linear logic". In: *Theoretical Computer Science* 294.1 (2003), pp. 183–231. ISSN: 0304-3975. DOI: https://doi.org/10.1016/S0304-3975(01)00241-9.

[Pis03] Gilles Pisier. *Introduction to Operator Space Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2003.

[BL04] David P. Blecher and Christian Le Merdy. *Operator Algebras and Their Modules: An operator space approach*. Oxford University Press, Oct. 2004. ISBN: 9780198526599. DOI: 10.1093/acprof:oso/9780198526599.001.0001.

[Gir04] Jean-Yves Girard. "Between Logic and Quantic: a Tract". In: *Linear Logic in Computer Science*. Ed. by Thomas Ehrhard et al. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004, pp. 346–381.

[Sel04] Peter Selinger. "Towards a semantics for higher-order quantum computation". In: *Proceedings of the 2nd International Workshop on Quantum Programming Languages, TUCS General Publication*. Vol. 33. 2004, pp. 127–143.

[Gir07a] Jean-Yves Girard. *Le point aveugle II: Cours de logique, Vers l'imperfection*. Visions des sciences. 2007. ISBN: 9782705666347.

[Gir07b] Jean-Yves Girard. "Truth, modality and intersubjectivity". In: *Mathematical Structures in Computer Science* 17.6 (2007), pp. 1153–1167. DOI: 10.1017/S0960129507006342.

[Bar10] Stefano Baratella. "Quantum coherent spaces and linear logic". In: *RAIRO-Theoretical Informatics and Applications* 44.4 (2010), pp. 419–441. DOI: 10.1051/ita/2010021.

[BPS10] Richard Blute, P. Panangaden, and Sergey Slavnov. "Deep Inference and Probabilistic Coherence Spaces". In: *Applied Categorical Structures* 20 (2010), pp. 209–228. DOI: 10.1007/s10485-010-9241-0.

[LZ24] Bert Lindenhovius and Vladimir Zamdzhiev. "The Category of Operator Spaces and Complete Contractions". In: *CoRR* abs/2412.20999 (2024). DOI: 10.48550/ARXIV.2412.20999. arXiv: 2412.20999.

[LZ25] Bert Lindenhovius and Vladimir Zamdzhiev. "Operator Spaces, Linear Logic and the Heisenberg-Schrödinger Duality of Quantum Theory". In: *2025 40th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. 2025, pp. 870–883. DOI: 10.1109/LICS65433.2025.00071. arXiv: 2505.06069.

[LZ26] Thea Li and Vladimir Zamdzhiev. *Quantum Coherence Spaces Revisited: A von Neumann (Co)Algebraic Approach*. 2026. arXiv: 2601.15832 [math.CT]. URL: https://arxiv.org/abs/2601.15832.