

Homological Algebra in Abelian Framed Bicategories

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Abstract

We introduce abelian framed bicategories, which are particular framed bicategories that are locally abelian, and show that they are suitable for developing homology and cohomology theories for directed structures. This means in particular that similar exact sequences as the relative homology and Mayer–Vietoris long exact sequences can be shown to hold. Also, for closed monoidal abelian framed bicategories, Künneth theorem holds as well. Finally, we prove embedding theorems similar to the Gabriel and Freyd–Mitchell theorems, for particular abelian framed bicategories, allowing to see those as bicategories of bimodules over algebras. This naturally links to the original motivation of this work, which was to generalize directed homology developed in the abelian framed bicategory of bimodules over (path) algebras.

Introduction Designing suitable homology and homotopy theories for directed objects, such as directed topological spaces, is a notorious difficult problem [5, 7, 4, 8, 1]. The non-reversibility of time in those models makes the needed algebra much more complex than in the classical case: to faithfully abstract this non-reversibility in time, the algebraic structures employed cannot be reversible (like abelian groups in classical algebraic topology, or ordered homology groups in [7]) or even cancellative (like Patchkoria’s theory in cancellative monoids [8]), because non-homotopic directed paths can become homotopic when extended, see Fahrenberg’s matchbox [4].

This means we have to move away from classical abelian categories as a suitable algebraic framework. Still, some form of local abelianity is needed: directed spaces are $(\infty, 1)$ -categories by nature (see [3]), because if paths are not invertible, (higher) homotopies are. Consequently, over the non-invertible low-dimensional data lies a more classical, fully invertible world where abelian structures can be used. That is the direction some took to develop non-cancellative homology theories for directed spaces, such as homology as functor from a category of traces to the singular homology of trace spaces for directed spaces [2] or homology bimodules over a path algebra for precubical sets [6]. Those theories have a similar flavor: they look at the classical homology modules of trace spaces between two points and describe how those modules evolve by pre- and post-composing with additional traces. Written differently, they make traces act on the left and on the right of an object in an abelian category. Following this idea, we introduce abelian framed bicategories as a new suitable framework for developing homology theories for directed structures.

Framed bicategories, introduced by Shulman in [9], are a way to categorify the way a structure, the coefficients, act (on the left and on the right) on another structure and how those actions evolve by changing the coefficients by extension and restriction. They are given by a bifibration $B \rightarrow C$ where C is the category of coefficients while B is a category of structures on which coefficients can act, and for which the effect of restriction (resp. the extension) of coefficients is given by cartesian (resp. cocartesian) liftings. Then, abelian framed bicategories are framed bicategories for which the fibers above pairs of coefficients (that is the subcategories of structures on which fixed coefficients act) are abelian and for which the bicategorical structure preserves enough of the abelian structure. The archetypal examples of such are categories of bimodules over algebras, such as the ones used in [6] to develop the homology of precubical sets.

Definition 0.1. An **abelian framed bicategory** is a framed bicategory \mathbb{D} that is locally abelian, i.e. all local categories $\mathcal{D}(A, B)$ are abelian, and such that horizontal composition yields additive functors $\mathcal{D}(A, B) \times \mathcal{D}(B, C) \rightarrow \mathcal{D}(A, C)$, $(M, N) \mapsto M \odot N$ for all objects $A, B, C \in \mathbb{D}_0$.

We define (co)chain complexes and (co)homology in an abelian framed bicategories \mathbb{A} .

Definition 0.2. A **chain complex** in \mathbb{A} is a chain complex in a local abelian category of \mathbb{A} , i.e., a chain complex in $\mathcal{A}(A, B)$ for some objects A and B . A **morphism** $(\alpha_i)_{i \geq 0}$ from a chain complex $(M_i)_{i \geq 0}$ in $\mathcal{A}(A, B)$ to a chain complex $(N_i)_{i \geq 0}$ in $\mathcal{A}(C, D)$ is a pair of vertical arrows $f : A \rightarrow C$ and $g : B \rightarrow D$ along with 2-cells $\alpha_i : M_i \xrightarrow[f]{g} N_i$ for all $i \geq 0$, such that $\alpha_i \partial_{i+1} = \partial_{i+1} \alpha_{i+1}$ for all $i \geq 0$.

Consider a functor $M : \mathcal{C} \rightarrow \text{Ch}(\mathbb{A})$ into the category of chain complexes in \mathbb{A} .

Definition 0.3. The **homology** of an object X in \mathcal{C} is the homology $H_*(X) : L(X) \rightarrow R(X)$ of the chain complex $M_*(X)$ in the local abelian category $\mathcal{D}(L(X), R(X))$. This construction is functorial.

As in abelian categories, we can derive interesting exact sequences. For particular morphisms $X \rightarrow Y$ in \mathcal{C} that we call relative pairs, there is a relative homology exact sequence, and a Mayer-Vietoris and Künneth theorems, involving extension of scalars:

Theorem 0.4. For all relative pairs (X, Y) , there is a long exact sequence for relative homology: $\cdots \rightarrow H_{i+1}(X, Y) \rightarrow {}^X H_i(Y) \rightarrow H_i(X) \rightarrow H_i(X, Y) \rightarrow \cdots$.

Finally, we show embedding theorems, similar to the Gabriel and Freyd–Mitchell theorems, for certain abelian framed bicategories with additional structure, that we call **module-like**.

Theorem 0.5. If \mathbb{A} is module-like, there is a framed lax functor $F : \mathbb{A} \rightarrow {}_R \text{Mod}_R$ into a concrete abelian framed bicategories of modules over R -algebras, with $R = \text{End}(U_I)$, such that F_1 is fully faithful and locally an equivalence of categories.

References

- [1] J. Dubut. *Directed homotopy and homology theories for geometric models of true concurrency*. PhD thesis, Université Paris-Saclay, 2017.
- [2] J. Dubut, E. Goubault, and J. Goubault-Larrecq. Natural homology. In *Automata, Languages, and Programming: 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part II 42*, pages 171–183. Springer, 2015.
- [3] J. Dubut, E. Goubault, and J. Goubault-Larrecq. The Directed Homotopy Hypothesis. In *Proceedings of the 25th Annual EACSL Conference on Computer Science Logic (CSL’16)*, volume 62 of *Leibniz International Proceedings in Informatics*, pages 1–16, 2016.
- [4] U. Fahrenberg. Directed homology. *Electronic Notes in Theoretical Computer Science*, 100:111–125, 2004.
- [5] E. Goubault. *Géométrie du parallélisme*. PhD thesis, Ecole Polytechnique, 1995.
- [6] E. Goubault. Directed homology and persistence modules. *Journal of Applied and Computational Topology*, 9(1):3, 2025.
- [7] M. Grandis. Inequillogical spaces, directed homology and noncommutative geometry. *Homology, Homotopy and Applications*, 6(1):413–437, 2004.
- [8] A. Patchkoria. On exactness of long exact sequences of homology semimodules. *Journal of Homotopy and Related structures*, 1(1):229–243, 2006.
- [9] M. A. Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–738, 2008.