

The Category \mathbf{Gm} for Extensive-Form Games

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This paper introduces \mathbf{Gm} , which is a category for extensive-form games. The category's objects are extensive-form games, each of which is regarded as a set of nodes which has been endowed with edges, information sets, actions, players, and utility functions. Its morphisms are functions from source nodes to target nodes that respect this structure. For instance, a game's information-set collection is newly regarded as a topological basis for the game's decision-node set, and then a morphism's continuity serves to respect the source game's information sets. Given these definitions, a game monomorphism is characterized by the property of not mapping two source runs (plays) to the same target run. Further, a game isomorphism is characterized as a bijection whose restriction to decision nodes is a homeomorphism, whose induced player transformation is bijective, and whose induced run transformation is monotonic with respect to the total preorder determined by each player's utility function.

1 Introduction

The abstract summarizes this paper's contributions. This introduction places the paper in the literature, and motivates it in the context of future research.

1.1 Placement in Literature

The large literature on game semantics uses specialized games to study logic and computation (some general references are [Hy197], [Jap03], and [AR24]). This literature borders on the present paper because it often uses category theory to describe the algebra of constructing games (see [Hy197] page 133). In contrast, the present paper uses category theory to compare games.

In a similar vein, there is a growing computer-science literature which composes extensive-form games from game fragments called "open games". Within this literature, the papers with nonsingleton information sets are [BHZ23] and [CGLNF22]. These papers define open games as morphisms, which are to be composed to create games. In contrast, the present paper defines games as objects, to be compared by morphisms.

So, closer to the categories here are categories in which the objects are specialized kinds of extensive-form games. [HLR12] builds a category for certain coalgebraic games. [LNS18] develops a category for certain extensive-form games on the basis of their plays. [Hed18] defines morphisms for some open games (previous paragraph) which have singleton information sets. [DST24] develops a category which provides a new proof of a topological equivalence. Finally, [BEE⁺24] and [BES25] develop categories for certain combinatoric games.

The author's research deviates from all the contributions in the previous paragraph by accommodating nonsingleton information sets (i.e., imperfect information). The author's first approach was restricted

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to node-and-choice games (i.e., distinguished-action games), which require that each information set has its own actions (see [Str26] Appendix A.3). Three papers resulted: [Str18] (for “preforms” resembling the CLTs in the present paper), [Str20a] (for “forms”), and finally [Str20b] (for games). The author’s second and current approach began with [Str21], which removed the distinguished-action restriction. [Str26] is a revision which more cleanly defines the concepts of out-tree and player transformation (and for the sake of brevity omits all results on subgames, equilibrium concepts, and subcategories).

The present paper abridges [Str26]. It has much less discussion, a quarter of the figures, and no proofs. Nonetheless, the section, equation, and proposition numbers here coincide with those of [Str26].

1.2 Motivation and Future Research

Category theory has added clarity and perspective to many fields of mathematics. So it seems valuable to do the same for the growing field of game theory.

The author’s own motivation is rooted in the field of economics. Specifically, the extensive-form games used by economists are formulated in many different styles (see the examples of [Str26] Appendix A). This diversity of style arises not only from historical accident, but also from the different notational needs of different mathematical contexts. Economists informally understand that [a] any “substantial” concept in one style should have the same meaning in any other style, and [b] any “substantial” result in one style should also hold in any other style. This informal understanding might be formally developed. In particular, when a concept or result is translated from one style to another, how would we define the sense in which the translation itself was correct or incorrect? A good answer to this question would help us to identify, manipulate, and communicate the “substance” of game theory. Category theory seems well-suited to this task.

Some progress has already been made. [Str21] showed that [1] the concepts of Nash equilibrium, subgame perfection, perfect information, and no-absentmindedness are each invariant to \mathbf{Gm} isomorphisms, [2] the full subcategory for distinguished-action games is essentially wide in \mathbf{Gm} , and [3] the full subcategory for choice-set games is essentially wide in the full subcategory for no-absentminded (i.e., linear) games. The latter two results are discussed by example in [Str26] Appendix A.4.

The present paper’s class of extensive-form games is broad enough to represent almost all extensive-form games studied by economists (exceptions such as continuous-time extensive-form games are discussed in [Str26] note 31). Beyond this are classes of closely related games formulated in very different ways. These include the open games and coalgebraic games mentioned in Section 1.1, and also the pentaform games of [Str25]. The author intends to build a category for the latter, and to show its relationship to \mathbf{Gm} .

2 Definition of CLT

2.1 Initial remarks

It is common to first define an extensive form, and then define an extensive-form game. Similarly, this Section 2 defines the category \mathbf{CLT} for “continuously labeled trees”, and then Section 3 defines the category \mathbf{Gm} for extensive-form games. The term “continuously labeled tree” (Definition 2.1) replaces “extensive form” because it is more descriptive and less ambiguous.

The particular game specification used here is discussed extensively in [Str26]. In developing this game specification, the author’s guiding principles were to avoid specialized assumptions, to use explicit notation, and to use standard mathematics whenever possible.

Out-tree (X, E)		(2)	2.2
X	set of nodes x		2.2
E	set of (directed) edges xy		2.2
r	↳ root node		2.2
W	↳ set of decision nodes x	(3)	2.2
p	↳ immediate predecessor function		2.2
\preceq	↳ weak tree order		2.2
\prec	↳ strict tree order		2.2
P	↳ strict predecessor correspondence		2.2
$X \setminus W$	↳ set of end nodes x		2.2
\mathcal{Z}	↳ collection of runs Z	(4)	2.2
\mathcal{Z}_{ft}	↳ collection of finite runs Z		2.2
$\mathcal{Z}_{\text{inft}}$	↳ collection of infinite runs Z		2.2
CLT (Continuously Labeled Tree) $\Theta = (X, E, \mathcal{H}, \lambda)$		(C1)–(C4)	2.3
\mathcal{H}	collection of information sets H		2.3
λ	labeling function (action-assigning function)		2.3
A	↳ set of actions a	(5)	2.3
F	↳ feasibility correspondence	(7)	2.3
n	↳ next-node function	(8)	2.3
CLT morphism $\theta = [\Theta, \Theta', \tau]$		(cE), (cH), (cL)	2.4
τ	node transformation		2.4
α	↳ action transformation	(11)	2.4
ζ	↳ run transformation (requires θ is end-preserving)	(19)	3.2
Game $\Gamma = (X, E, \mathcal{H}, \lambda, \kappa, U)$		(G1)–(G3)	3.1
κ	control-assigning function (player-assigning function)		3.1
I	↳ set of players i	(15)	3.1
U	profile $\langle U_i \rangle_{i \in I}$ listing utility function U_i for each player i		3.1
\lesssim_i	↳ total preorder of \mathcal{Z} implied by U_i	(17)	3.1
Game morphism $\gamma = [\Gamma, \Gamma', \tau]$		(gZ), (gK), (gU)	3.2
ι	↳ player transformation	(21)	3.2

Table 1: Terms defined in Sections 2 and 3. Out-trees, CLTs, CLT morphisms, games, and game morphisms are implicitly accompanied by their components and derivatives (↳). Definitions are in equations and sections on right side.

2.2 Out-trees

Before defining out-trees, define a *directed graph* to be a couple (X, E) such that $E \subseteq X^2$. The elements of X are called *nodes*. The elements of E are called *edges*, and are denoted by xy rather than $\langle x, y \rangle$. A *walk from w to y in a directed graph (X, E)* is a sequence $\langle x_0, x_1, x_2, \dots, x_m \rangle$ such that

$$x_0 = w, x_m = y, \text{ and } (\forall \ell < m) x_\ell x_{\ell+1} \in E. \quad (1)$$

Similarly, an *infinite walk from w in a directed graph (X, E)* is a sequence $\langle x_0, x_1, x_2, \dots \rangle$ such that $x_0 = w$ and $(\forall \ell) x_\ell x_{\ell+1} \in E$. Note that a walk in an arbitrary directed graph can visit a node more than once.

An *out-tree*^[1] is a directed graph (X, E) such that

$$(\exists r \in X)(\forall x \in X) \text{ there exists a unique walk from } r \text{ to } x \text{ in } (X, E). \quad (2)$$

It can be shown that each out-tree has exactly one r that satisfies (2). Call this the *root node* of the out-tree. For example, $(\{5\}, \emptyset)$ is a one-node out-tree because the one-node sequence $\langle 5 \rangle$ is the unique walk from 5 to itself in $(\{5\}, \emptyset)$. In this example, 5 is the root node r . Such one-node out-trees are the “smallest” out-trees because (\emptyset, \emptyset) is not an out-tree (it lacks the node r required by (2)). In the opposite direction, the “largest” out-trees can have uncountably infinite degree (they can have nodes with uncountably many immediate successors) and countably infinite height (they can have infinite walks of the sort defined below (1)).

Consider an out-tree (X, E) . Let

$$W = \pi_1 E, \quad (3)$$

where $\pi_1 E$ is the projection of E onto its first coordinate. Call W the *decision-node* set. It can be shown that $E^{-1} = \{ \langle y, x \rangle \mid xy \in E \}$ is the graph of a surjective function^[2] from the set $X \setminus \{r\}$ of nonroot nodes to the decision-node set W . Let p denote this function, and call it the *immediate-predecessor function*. (If the out-tree (X, E) is of the extreme form $(\{x\}, \emptyset)$, then $W, X \setminus \{r\}$, and p are all empty.)

Further, let \preceq be the binary relation on X defined by $x \preceq y$ iff there is a walk from x to y . Call \preceq the *weak tree order*. Note \preceq is reflexive because of the one-node walk from any node to itself. Next, let \prec be the binary relation on X defined by $x \prec y$ iff $x \preceq y$ and $x \neq y$. Call \prec the *strict tree order*. It can be shown that \preceq is a partial order on X and that \prec is its asymmetric part. Further, define the correspondence (note [2]) $P: X \rightrightarrows W$ by $P(y) = \{x \in X \mid x \prec y\}$. Call P the *strict predecessor correspondence*. It is easily shown that $(\forall y \in X \setminus \{r\}) p(y) \in P(y)$.

It can be shown that each walk in the out-tree (X, E) visits each of its nodes exactly once^[3] and that each walk in (X, E) is characterized by the set of nodes that it visits. Because of this, each walk in (X, E) will be identified with the set of nodes that it visits. For example, for each node $y \in X$, the set $P(y) \cup \{y\}$ is the walk from the root node r to the node y (expressed as a set rather than a sequence).

Further, call $X \setminus W$ the set of *end nodes*, and let \mathcal{L}_{ft} be the collection of all walks from r to an end node (each walk is expressed as a set $Z \subseteq X$). Further, let \mathcal{L}_{inf} be the collection of all infinite walks from r (as defined below (1)), but each walk is expressed as a set $Z \subseteq X$). Finally, let

$$\mathcal{L} = \mathcal{L}_{\text{ft}} \cup \mathcal{L}_{\text{inf}}, \quad (4)$$

and call \mathcal{L} the collection of *runs* (elsewhere “plays”). Note that \mathcal{L} is nonempty (if the out-tree (X, E) is of the extreme form $(\{x\}, \emptyset)$, then the end-node set is $X \setminus W = \{x\}$ and the only run in \mathcal{L} is $Z = \{x\}$). Possibly $\mathcal{L} = \mathcal{L}_{\text{ft}}$, possibly $\mathcal{L} = \mathcal{L}_{\text{inf}}$, and possibly both \mathcal{L}_{ft} and \mathcal{L}_{inf} are nonempty.

¹Similar definitions elsewhere include include [GS53], [KL16], [Str18], [GDGJFJ23], [Tut84], and [BJG09]. Several of these definitions only apply to finite trees, and several definitions explicitly include antisymmetry or irreflexivity. The definition here applies equally well to infinite trees, and implies antisymmetry and irreflexivity. The term “out-tree” [BJG09] is used instead of “outwardly directed rooted tree” and “diverging arborescence” only because it is shorter.

²To be clear, a *correspondence* $F: X \rightrightarrows Y$ is taken to be a triple $[X, Y, F^{\text{gr}}]$ such that $F^{\text{gr}} \subseteq X \times Y$, and a *function* $f: X \rightarrow Y$ is taken to be a triple $[X, Y, f^{\text{gr}}]$ such that $f^{\text{gr}} \subseteq X \times Y$ and $(\forall x \in X)(\exists! y \in Y) (x, y) \in f^{\text{gr}}$. The term “correspondence” is standard in economics but perhaps not elsewhere.

³This implies that each walk in (X, E) is a “path”, where a “path” is a special kind of walk which visits each of its nodes exactly once. This paper does not use the term “path”.

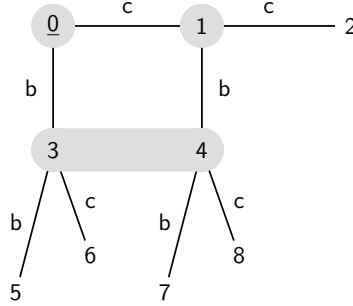


Figure 1: A continuously labeled tree (CLT). A CLT consists of an out-tree (X, E) , a partition \mathcal{H} of the tree’s decision-node set W , and a labeling (action-assigning) function λ .

2.3 Continuously labeled trees (CLTs)

Consider an out-tree (X, E) . Let \mathcal{H} be a partition of the tree’s decision-node set W (3), and call its elements *information sets*. For example, consider Figure 1. There the decision-node set is $W = \{0, 1, 3, 4\}$, and the shaded regions depict that W is partitioned by the information sets belonging to the collection $\mathcal{H} = \{\{0\}, \{1\}, \{3, 4\}\}$.

One can regard \mathcal{H} as the basis of a topology for W . Call this topology the *information topology* for W . Thus a set of decision nodes is open in the information topology for W iff it is the union of a collection of information sets from \mathcal{H} . This paper uses the information topology for W and the discrete topology everywhere else.

Next, let λ be a surjective function from the edge set E to a discrete (i.e. unstructured) codomain. Call λ the *labeling function* (elsewhere an “action-assigning function”), and say that $\lambda(xy)$ *labels* the edge xy . Next define A to be the codomain of λ . In other words (since λ is surjective), let⁴

$$A = \bar{\lambda}(E), \quad (5)$$

where $\bar{\lambda}(E) = \{ \lambda(xy) \mid xy \in E \}$ denotes the image of the set E . Although A could be called the set of “labels”, A will instead be called the set of *actions*. For example, the action set A is $\{b, c\}$ in Figure 1.

It will be assumed that λ is *locally injective* in the sense that, for any two edges of the form $xy_1 \in E$ and $xy_2 \in E$,

$$y_1 \neq y_2 \text{ implies } \lambda(xy_1) \neq \lambda(xy_2). \quad (6)$$

This implies that the edges leaving each decision node $x \in W$ are injectively labeled. For example, the labeling function λ shown in Figure 1 is locally injective even though it is not injective. Local injectivity is equivalent to the concept of “determinism” in [BRV01] page 3.

From λ derive the correspondence $F: W \rightrightarrows A$ by

$$(\forall x \in W) F(x) = \{ a \in A \mid (\exists y) \lambda(xy) = a \}. \quad (7)$$

Thus $F(x)$ is the set of actions that label the edges leaving x . Call F the *feasibility correspondence*, and call $F(x)$ the set of actions that are *feasible* at x . For example, $F(3) = \{b, c\}$ in Figure 1.

⁴To be clear, recall from note 2 that a function is regarded as a triple. Here, $\lambda = [\lambda^{\text{dom}}, \lambda^{\text{cod}}, \lambda^{\text{gr}}]$ is assumed to be a surjective function from E . This implies $\lambda^{\text{dom}} = E$ and $\lambda^{\text{cod}} = \bar{\lambda}(E)$. Next, A is defined to be $\lambda^{\text{cod}} = \bar{\lambda}(E)$. Since A is derived from λ , A will not appear in the tuple defining a CLT. Accordingly, Table 1 lists λ as a component of a CLT, and A as a derivative of a CLT. (Incidentally, the assumption of surjectivity avoids redundant λ ’s which differ only in their codomains.)

For future reference, we define a function n which takes each decision node $x \in W$ and each feasible action $a \in F(x)$ to the node $n(x, a)$ at the end of the edge from x labeled by a . Casually, $n(x, a)$ is the result of choosing a at x . Formally, assume λ is locally injective and define⁵ $n: F^{\text{gr}} \rightarrow X \setminus \{r\}$ implicitly by⁶

$$(\forall (x, a) \in F^{\text{gr}}) \lambda(xn(x, a)) = a. \quad (8)$$

Thus $n(x, a)$ is defined to be the unique y_* such that $\lambda(xy_*) = a$. Call n the *next-node function* (it is well-defined by [Str26] Lemma B.1(b)). For example, $n(3, c) = 6$ in Figure 1.

From a different perspective, $F = \langle F(x) \rangle_{x \in W}$ can be regarded as a set-valued function from W to $\mathcal{P}(A)$. It will be assumed that F is continuous as a function from W , endowed with the information topology, to $\mathcal{P}(A)$, endowed with the discrete topology. This is equivalent to

$$(\forall H \in \mathcal{H}, x_1 \in H, x_2 \in H) F(x_1) = F(x_2), \quad (9)$$

which requires that two nodes in one information set have the same feasible set. For example, in Figure 1, nodes 3 and 4 share an information set, and in accord with (9), the nodes share the feasible set $F(3) = F(4) = \{b, c\}$.

Definition 2.1 A *continuously labeled tree (CLT)* is a tuple $\Theta = (X, E, \mathcal{H}, \lambda)$ such that

$$(X, E) \text{ is an out-tree (2)}, \quad (C1)$$

$$\mathcal{H} \text{ is a partition of } W, \quad (C2)$$

$$\lambda \text{ is a locally injective (6) surjective function from } E, \text{ and} \quad (C3)$$

$$\langle F(x) \rangle_{x \in W} \text{ is continuous (9) from } W \quad (C4)$$

(where E determines W by (3) and λ determines F by (7)).

Note that a tree diagram with underlined root node, shaded information sets, and labeled edges unambiguously specifies all the components of a CLT. Thus Figure 1 defines a CLT.

2.4 CLT morphisms

This Section 2.4 will define CLT morphisms. This first paragraph fixes some general notation. For $f: X \rightarrow Y$ and $A \subseteq X$, let $\bar{f}(A) = \{f(x) \mid x \in A\}$. Also, for $f: X \rightarrow Y$, $A \subseteq X$, and $B \subseteq Y$, let $f|_{A, B}$ be the function with domain A , codomain B , and the graph of $f|_A$. This construction is well-defined iff $\bar{f}(A) \subseteq B$.

Consider a source CLT Θ and a target CLT Θ' . A *node transformation* is a function of the form $\tau: X \rightarrow X'$. Definition 2.4 will assume that τ *preserves edges* in the sense that

$$(\forall xy \in E) \tau(x)\tau(y) \in E'. \quad (10)$$

Edge preservation implies that $[(X, E), (X', E'), \tau]$ is a graph morphism, and easily leads to the several consequences in Proposition 2.5(a-e). Among these consequences is $\bar{\tau}(W) \subseteq W'$, which implies that $\tau|_{W, W'}$ is well-defined (recall the general notation of the previous paragraph).

⁵ F^{gr} is the graph of the correspondence F , as in note 2. Notice that $(x, a) \in F^{\text{gr}}$ iff $a \in F(x)$.

⁶ [Str26] Lemma B.1(c) shows that $\lambda(x, \cdot)$ and $n(x, \cdot)$ are “local inverses” at each decision node x . Essentially, $\lambda(x, \cdot)$ maps immediate successors y to feasible actions a , and conversely, $n(x, \cdot)$ maps feasible actions a to immediate successors y .

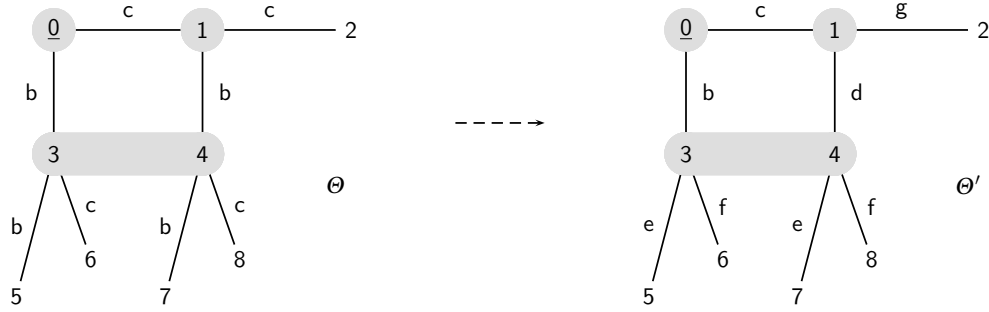


Figure 2: The CLT Θ (from Figure 1) and the CLT Θ' have different actions. The tuple $[\Theta, \Theta', \text{id}_X]$ is a morphism.

Definition 2.4 will also assume that $\tau|_{W, W'}$ is continuous. As would be expected from general topology, this means that the inverse image of every target information set is the union of a collection of source information sets. However, Proposition 2.2 provides another perspective by taking advantage of the fact that W and W' are endowed with partition topologies. The proposition shows that continuity is equivalent to the image of each source information set being included in a target information set. In other words, continuity says that source information sets cannot be “split”.

Proposition 2.2 *Suppose Θ and Θ' are CLTs, $\tau: X \rightarrow X'$, and $\bar{\tau}(W) \subseteq W'$. Then $\tau|_{W, W'}$ is continuous iff $(\forall H \in \mathcal{H})(\exists H' \in \mathcal{H}') \bar{\tau}(H) \subseteq H'$. ([Str26] Proof B.2)*

To interpret continuity in the context of game theory, we only need two observations: [1] that large information sets correspond to less information, and [2] that continuity prevents information sets from being split. Thus, continuity corresponds to preserving a lack of information.

Next, from the tuple $[\Theta, \Theta', \tau]$, derive $\alpha = \langle \alpha_x: F(x) \rightarrow F'(\tau(x)) \rangle_{x \in W}$ at each decision node $x \in W$ by

$$(\forall a \in F(x)) \alpha_x(a) = \lambda'(\tau(x) \tau(n(x, a))) \quad (11)$$

(where the source labeling function λ determines the source next-node function n by (8)). Call $\alpha = \langle \alpha_x \rangle_{x \in W}$ the tuple’s *action transformation*. At each $x \in W$, the statement $\alpha_x: F(x) \rightarrow F'(\tau(x))$ means that the function α_x sends source actions feasible at x to target actions feasible at the image of x . In particular, consider a source action a which is feasible at x . By (11), this a is sent to the target action that labels the target edge from the image of x to the image of the next source node determined by x and a . For example in Figure 2, equation (11) implies $\alpha_3(b)$ is $\lambda'(\tau(3) \tau(n(3, b)))$, which reduces to $\lambda'(\tau(3) \tau(5)) = \lambda'(35) = e$.

Proposition 2.3 *Suppose Θ and Θ' are CLTs, and $\tau: X \rightarrow X'$ preserves edges (10). Then the action transformation $\alpha = \langle \alpha_x \rangle_{x \in W}$ (11) is well-defined. ([Str26] Proof B.3)*

As a whole, $\alpha = \langle \alpha_x \rangle_{x \in W}$ is a function-valued function with domain W . Definition 2.4 will assume that α is continuous as a function from W , endowed with the information topology, to the set of functions, endowed with the discrete topology. This is equivalent to assuming that

$$(\forall H \in \mathcal{H}, x_1 \in H, x_2 \in H) \alpha_{x_1} = \alpha_{x_2}. \quad (12)$$

Conveniently, [Str26] note 16 shows that if $\tau|_{W, W'}$ is continuous, then (12) is equivalent to

$$(\forall H \in \mathcal{H}, x_1 \in H, x_2 \in H) (a \in F(x_1)) \alpha_{x_1}(a) = \alpha_{x_2}(a). \quad (13)$$

Thus the continuity of α is equivalent to requiring that two nodes in the same information set transform actions in the same way.

Note that the continuity of α from W allows “local” action changes in the sense that actions can be changed differently at different information sets. For example, $\alpha_0(b) = b$ and $\alpha_3(b) = e$ in Figure 2. This difference is consistent with the continuity of α (12) because 0 and 3 are in different information sets.

Definition 2.4 A **CLT morphism** is a tuple $\theta = [\Theta, \Theta', \tau]$ such that Θ and Θ' are CLTs (Definition 2.1), $\tau: X \rightarrow X'$,

$$\tau \text{ is edge-preserving (10),} \quad (\text{cE})$$

$$\tau|_{W, W'} \text{ is continuous, and} \quad (\text{cH})$$

$$\langle \alpha_x \rangle_{x \in W} \text{ is continuous (12) from } W \quad (\text{cL})$$

(where $[\Theta, \Theta', \tau]$ determines $\langle \alpha_x \rangle_{x \in W}$ by (11)).

Mnemonicly, (cE) abbreviates “edge preservation”, (cH) “information-set preservation”, and (cL) “label preservation” (the leading “c” in (cE), (cH), and (cL) signifies “CLT”). Note that the source root r may or may not be taken to the target root r' .

Proposition 2.5 Suppose $[\Theta, \Theta', \tau]$ is a morphism. Then the following hold.

- (a) $(\forall x \in X, y \in X) x \preccurlyeq y \Rightarrow \tau(x) \preccurlyeq' \tau(y)$.
- (b) $(\forall x \in X, y \in X) x \prec y \Rightarrow \tau(x) \prec' \tau(y)$.
- (c) $(\forall y \in X) \bar{\tau}(P(y)) \subseteq P'(\tau(y))$.
- (d) $(\forall y \in X \setminus \{r\}) \tau(p(y)) = p'(\tau(y))$.
- (e) $\bar{\tau}(W) \subseteq W'$.
- (f) $(\forall xy \in E) \alpha_x(\lambda(xy)) = \lambda'(\tau(x)\tau(y))$.
- (g) $(\forall x \in W, a \in F(x)) \tau(n(x, a)) = n'(\tau(x), \alpha_x(a))$. (*[Str26] Proof B.4*)

2.5 The category CLT

This paragraph defines the category **CLT**, which is called the *category of CLTs (continuously labeled trees)*. Let an object be a CLT Θ (Definition 2.1). Let an arrow be a CLT morphism $[\Theta, \Theta', \tau]$ (Definition 2.4). Let source, target, identity, and composition be

$$\begin{aligned} [\Theta, \Theta', \tau]^{\text{src}} &= \Theta, [\Theta, \Theta', \tau]^{\text{trg}} = \Theta', \text{id}_\Theta = [\Theta, \Theta, \text{id}_X], \text{ and} \\ &[\Theta', \Theta'', \tau'] \circ [\Theta, \Theta', \tau] = [\Theta, \Theta'', \tau' \circ \tau] \end{aligned} \quad (14)$$

(where id_X is an identity in **Set**).

Proposition 2.6

- (a) **CLT** is well-defined.
- (b) If Θ is a CLT, then the action transformation of id_Θ is $\langle \text{id}_{F(x)} \rangle_{x \in W}$.
- (c) If $[\Theta, \Theta', \tau]$ and $[\Theta', \Theta'', \tau']$ are morphisms, then the action transformation of $[\Theta', \Theta'', \tau'] \circ [\Theta, \Theta', \tau]$ is $\langle \alpha'_{\tau(x)} \circ \alpha_x \rangle_{x \in W}$. (*[Str26] Proof B.7*)

3 Definition of Gm

3.1 Games

To define a game, begin with a CLT $\Theta = (X, E, \mathcal{H}, \lambda)$. Then let κ be a surjective function from the decision-node set W to a discrete (i.e. unstructured) codomain. Define I to be the codomain of κ . In other words (since κ is surjective), let

$$I = \bar{\kappa}(W). \quad (15)$$

Call κ the *control-assigning function* (elsewhere the “player-assigning function” or “move-assigning function”), call a member of I a *player*, and say that player $\kappa(x)$ *controls* node x . For example, consider Figure 4’s left-hand side. There, player PI controls the decision nodes 0 and 1 (each of these nodes happens to have a singleton feasible set). In this case, $W = \{0, 1\}$, $\kappa(0) = \kappa(1) = PI$, and $I = \{PI\}$.

Definition 3.1 will assume that κ is continuous. This means that, for each $i \in I$, the inverse image $\{x \in W \mid \kappa(x) = i\}$ is open in the information topology. This is equivalent to saying that, for each $i \in I$, $\{x \in W \mid \kappa(x) = i\}$ is the union of a collection of information sets. Equivalently,

$$(\forall H \in \mathcal{H}, x_1 \in H, x_2 \in H) \kappa(x_1) = \kappa(x_2), \quad (16)$$

which requires that two nodes in one information set are assigned to the same player.

Next, for each $i \in I$, consider a function $U_i: \mathcal{Z} \rightarrow \bar{\mathbb{R}}$, where $\bar{\mathbb{R}}$ denotes the set of extended real numbers, that is, $\mathbb{R} \cup \{-\infty, \infty\}$. Call U_i the *utility function* of player i (elsewhere the “payoff function” of player i). From the function $U_i: \mathcal{Z} \rightarrow \bar{\mathbb{R}}$, derive the total preorder \lesssim_i on \mathcal{Z} defined by

$$(\forall Z_1 \in \mathcal{Z}, Z_2 \in \mathcal{Z}) Z_1 \lesssim_i Z_2 \text{ iff } U_i(Z_1) \leq U_i(Z_2). \quad (17)$$

Definition 3.1 An **(extensive-form) game** is a tuple $\Gamma = (X, E, \mathcal{H}, \lambda, \kappa, U)$ such that

$$(X, E, \mathcal{H}, \lambda) \text{ is a CLT (Definition 2.1),} \quad (G1)$$

$$\kappa \text{ is a continuous (16) surjective function from } W, \text{ and} \quad (G2)$$

$$U \text{ is of the form } \langle U_i: \mathcal{Z} \rightarrow \bar{\mathbb{R}} \rangle_{i \in I} \quad (G3)$$

(where (X, E) determines W and \mathcal{Z} by (3) and (4), and I is κ ’s codomain (15)).

Figure 4 shows two games called Γ^1 and Γ^2 . More interesting examples are in [Str26].

3.2 Game morphisms

Game morphisms will be built on certain CLT morphisms. Say that a CLT morphism $[\Theta, \Theta', \tau]$ *preserves ends* iff

$$\bar{\tau}(X \setminus W) \subseteq X' \setminus W'. \quad (18)$$

Figure 3 shows a morphism which does not preserve ends, because the source end node 2 is mapped to the target decision node 12. Accordingly, the source run $\{0, 2\}$ is mapped to the non-run walk $\{10, 12\}$. Arguably, the source utility assigned to the source run $\{0, 2\}$ should imply nothing about the target utilities assigned to the target runs $\{10, 12, 13\}$ and $\{10, 12, 14\}$ because the connection between that

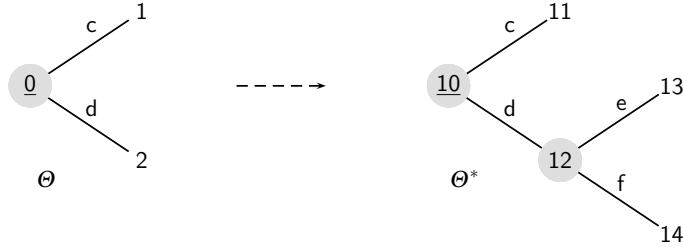


Figure 3: The CLT morphism $[\Theta, \Theta^*, \tau]$ defined by $\tau(x) = x + 10$. This morphism is not end-preserving (18).

source run and those two target runs is so nebulous. For this reason, game morphisms are not built on CLT morphisms that fail to preserve ends.

Rather, from an end-preserving $[\Theta, \Theta', \tau]$, derive the function $\zeta: \mathcal{Z} \rightarrow \mathcal{Z}'$ by

$$\zeta(Z) = P' \circ \tau(r) \cup \bar{\tau}(Z). \quad (19)$$

Call ζ the *run transformation*. In English, ζ maps a source run Z to its image $\bar{\tau}(Z)$ and then prepends $\bar{\tau}(Z)$ with the walk from r' to the image $\tau(r)$. Note $\tau(r)$ is also in $\bar{\tau}(Z)$ since $r \in Z$ because Z is a run.

Proposition 3.2 *If a morphism $[\Theta, \Theta', \tau]$ is end-preserving (18), then its run transformation ζ (19) is well-defined. ([Str26] Proof C.1)*

Next, consider two games Γ and Γ' together with a run transformation $\tau: X \rightarrow X'$. The triple $[\Gamma, \Gamma', \tau]$ is said to *preserve control* iff⁷

$$(\forall x_1 \in W, x_2 \in W) \kappa(x_1) = \kappa(x_2) \text{ implies } \kappa'(\tau(x_1)) = \kappa'(\tau(x_2)). \quad (20)$$

This requires that any two source nodes controlled by one player must be sent to target nodes controlled by one player. Figure 4 shows an example which violates this assumption.

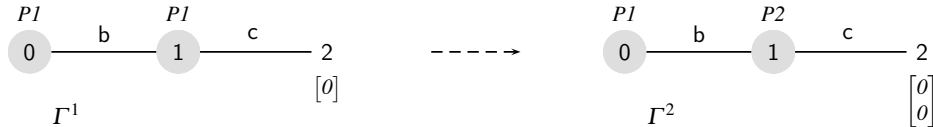


Figure 4: The games Γ^1 and Γ^2 have different players. The tuple $[\Gamma^1, \Gamma^2, \text{id}_{X^1}]$ is not a game morphism since it is not control-preserving (20).

From a control-preserving tuple $[\Gamma, \Gamma', \tau]$, derive the function $\iota: I \rightarrow I'$ with graph

$$\iota^{\text{gr}} = \{ (\kappa(x), \kappa'(\tau(x))) \mid x \in W \}. \quad (21)$$

Call ι the *player transformation* of $[\Gamma, \Gamma', \tau]$. Equation (21) says that ι maps the player controlling decision node x to the player controlling the image of x .

Proposition 3.3 *Suppose that Γ and Γ' are games, that $\tau: X \rightarrow X'$, and that $[\Gamma, \Gamma', \tau]$ is a control-preserving (20) tuple. Then (a) the player transformation ι (21) is well-defined, and (b) $\iota \circ \kappa = \kappa' \circ \tau|_{W, W}$. ([Str26] Proof C.2)*

⁷In (20)'s concluding equation, each side is well-defined because $(\forall w \in W) \tau(w) \in W'$ by Proposition 2.5(e).

Definition 3.4 A **game morphism** is a tuple $\gamma = [\Gamma, \Gamma', \tau]$ such that Γ and Γ' are games,

$$[\Theta, \Theta', \tau] \text{ is an end-preserving (18) CLT morphism (Definition 2.4),} \quad (\text{gZ})$$

$$[\Gamma, \Gamma', \tau] \text{ is control-preserving (20), and} \quad (\text{gK})$$

$$(\forall i \in I) \zeta \text{ is monotone from } (\mathcal{Z}, \lesssim_i) \text{ to } (\mathcal{Z}', \lesssim'_{i(i)}) \quad (\text{gU})$$

(where $[\Gamma, \Gamma', \tau]$ determines ζ and ι by (19) and (21)).

To summarize briefly, (gZ) explicitly preserves ends, and implicitly preserves edges, information sets and labels via the Definition 2.4 of a CLT morphism. Then (gK) preserves control. Finally, the definition (17) of \lesssim_i implies that (gU) is equivalent to

$$(\forall i \in I, Z_1 \in \mathcal{Z}, Z_2 \in \mathcal{Z}) U_i(Z_1) \leq U_i(Z_2) \text{ implies } U_{\iota(i)}(\zeta(Z_1)) \leq U_{\iota(i)}(\zeta(Z_2)). \quad (22)$$

3.3 The category **Gm**

This paragraph defines the category **Gm**, which is called the *category of extensive-form games*. Let an object be a game Γ (Definition 3.1). Let an arrow be a game morphism $[\Gamma, \Gamma', \tau]$ (Definition 3.4). Let source, target, identity, and composition be

$$\begin{aligned} [\Gamma, \Gamma', \tau]^{\text{src}} &= \Gamma, \quad [\Gamma, \Gamma', \tau]^{\text{trg}} = \Gamma', \quad \text{id}_\Gamma = [\Gamma, \Gamma, \text{id}_X], \text{ and} \\ [\Gamma', \Gamma'', \tau'] \circ [\Gamma, \Gamma', \tau] &= [\Gamma, \Gamma'', \tau' \circ \tau]. \end{aligned} \quad (23)$$

Proposition 3.5

(a) **Gm** is well-defined.

(b) Suppose Γ is a game. Then the run transformation of id_Γ is $\text{id}_\mathcal{Z}$, and its player transformation is id_I .

(c) Suppose $[\Gamma, \Gamma', \tau]$ and $[\Gamma', \Gamma'', \tau']$ are morphisms. Then the run transformation of $[\Gamma', \Gamma'', \tau'] \circ [\Gamma, \Gamma', \tau]$ is $\zeta' \circ \zeta$, and its player transformation is $\iota' \circ \iota$. ([Str26] Proof C.6.)

Proposition 3.6 Define **F** from **Gm** to **CLT** by

$$\begin{aligned} F_0 : (X, E, \mathcal{H}, \lambda, \kappa, U) &\mapsto (X, E, \mathcal{H}, \lambda) \text{ and} \\ F_1 : [\Gamma, \Gamma', \tau] &\mapsto [F_0(\Gamma), F_0(\Gamma'), \tau]. \end{aligned}$$

Then **F** is a well-defined functor. ([Str26] Proof C.7.)

The “forgetful” functor of Proposition 3.6 allows Section 2’s results for CLTs to be applied to games in a natural but rigorous way. For example, Propositions 3.6 and 2.6(c) imply that if γ and γ' are compatible game morphisms, then the action transformation of $\gamma' \circ \gamma$ is $\langle \alpha'_{\tau(x)} \circ \alpha(x) \rangle_{x \in W}$.

4 Special Morphisms

4.1 Monomorphisms

Proposition 4.1 A CLT morphism $[\Theta, \Theta', \tau]$ is monic iff τ is injective. ([Str26] Proof D.1.)

Proposition 4.2 A **Gm** morphism $[\Gamma, \Gamma', \tau]$ is monic iff its run transformation ζ is injective. ([Str26] Proof D.6.)

By ζ ’s definition (19), the injectivity of ζ in Proposition 4.2 is weaker than the injectivity of τ in Proposition 4.1, and further, [Str26] Section 4.1 shows by example that it is strictly weaker. The same section argues that the salient difference between Propositions 4.1 and 4.2 is that game morphisms are built on CLT morphisms that are end-preserving (Definition 3.4 condition (gZ)).

4.2 Isomorphisms

Proposition 4.3 uses the term “homeomorphism” from topology. Easily, if $[\Theta, \Theta', \tau]$ is a morphism and τ is bijective, then $\tau|_{W, W'}$ is a homeomorphism iff $(\tau|_{W, W'})^{-1}$ is continuous. Further, [Str26] Lemma E.2 shows that if $[\Theta, \Theta', \tau]$ is a morphism and τ is bijective, then $\tau|_{W, W'}$ is a homeomorphism iff $\mathcal{H} \ni H \mapsto \bar{\tau}(H) \in \mathcal{H}'$ is a bijection. Thus a homeomorphism does not split information sets in either direction.

Proposition 4.3 A **CLT** morphism $[\Theta, \Theta', \tau]$ is an isomorphism iff

- (a) τ is a bijection⁸ and
 - (b) $\tau|_{W, W'}$ is a homeomorphism.
- Further, if $[\Theta, \Theta', \tau]$ is an isomorphism, the following hold.
- (c) $\tau|_{X \setminus W, X' \setminus W'}$ is a bijection.
 - (d) $E \ni xy \mapsto \tau(x)\tau(y) \in E'$ is a bijection.
 - (e) $(\forall x \in X, y \in X) x \preceq y \Leftrightarrow \tau(x) \preceq' \tau(y)$.
 - (f) $(\forall x \in X, y \in X) x \prec y \Leftrightarrow \tau(x) \prec' \tau(y)$.
 - (g) $\tau(r) = r'$.
 - (h) $(\forall y \in X) \bar{\tau}(P(y)) = P'(\tau(y))$.
 - (i) $\bar{\tau}|_{\mathcal{H}, \mathcal{H}'}$ is a bijection.
 - (j) In the action transformation $\langle \alpha_x \rangle_{x \in W}$ of $[\Theta, \Theta', \tau]$, each α_x is a bijection.
 - (k) $[\Theta, \Theta', \tau]$ is end-preserving.
 - (l) The run transformation ζ of $[\Theta, \Theta', \tau]$ is the bijection $\bar{\tau}|_{\mathcal{Z}, \mathcal{Z}'}$.
 - (m) The inverse of $[\Theta, \Theta', \tau]$ is $[\Theta', \Theta, \tau^{-1}]$.
 - (n) The action transformation of $[\Theta', \Theta, \tau^{-1}]$ is $\langle \alpha_{\tau^{-1}(x')} \rangle_{x' \in W'}$.
 - (o) $[\Theta', \Theta, \tau^{-1}]$ is end-preserving.
 - (p) The run transformation of $[\Theta', \Theta, \tau^{-1}]$ is ζ^{-1} . ([Str26] Proof E.5)

Proposition 4.4(c) below uses the term “preorder isomorphism”. This is the concept of order isomorphism from order theory, extended from orders to preorders ([DP02] page 3; [Vig23]). In particular, if $[\Theta, \Theta', \tau]$ is a **CLT** isomorphism, then Proposition 4.3(l) implies $\zeta: \mathcal{Z} \rightarrow \mathcal{Z}'$ is a bijection, and thus for each player $i \in I$, ζ is a preorder isomorphism from (\mathcal{Z}, \preceq_i) to $(\mathcal{Z}', \preceq'_{i(i)})$ iff both ζ and ζ^{-1} are monotone.

Proposition 4.4 A **Gm** morphism $[\Gamma, \Gamma', \tau]$ is an isomorphism iff

- (a) $[\Theta, \Theta', \tau]$ is a **CLT** isomorphism,
- (b) ι is a bijection,
- (c) $(\forall i \in I) \zeta$ is a preorder isomorphism from (\mathcal{Z}, \preceq_i) to $(\mathcal{Z}', \preceq'_{i(i)})$.

Further, if $[\Gamma, \Gamma', \tau]$ is an isomorphism, the following hold.

- (d) The inverse of $[\Gamma, \Gamma', \tau]$ is $[\Gamma', \Gamma, \tau^{-1}]$.
- (e) The player transformation of $[\Gamma', \Gamma, \tau^{-1}]$ is ι^{-1} . ([Str26] Proof E.8)

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⁸Condition (a) follows from condition (b) when neither Θ nor Θ' has an end node, that is, when $X \setminus W = \emptyset$ and $X' \setminus W' = \emptyset$. This happens when both Θ and Θ' have only infinite runs.

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THE CATEGORY \mathbf{Gm} FOR EXTENSIVE-FORM GAMES (full version)

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ABSTRACT. This paper introduces \mathbf{Gm} , which is a category for extensive-form games. The category's objects are extensive-form games, each of which is regarded as a set of nodes which has been endowed with edges, information sets, actions, players, and utility functions. Its arrows are functions from source nodes to target nodes that respect this structure. For instance, a game's information-set collection is newly regarded as a topological basis for the game's decision-node set, and then a morphism's continuity serves to respect the source game's information sets. Given these definitions, a game monomorphism is characterized by the property of not mapping two source runs (plays) to the same target run. Further, a game isomorphism is characterized as a bijection whose restriction to decision nodes is a homeomorphism, whose induced player transformation is bijective, and whose induced run transformation is monotonic with respect to the total preorder determined by each player's utility function.

1. INTRODUCTION

Abstractly, an extensive-form game is a set with structure. More precisely, a game is a set of nodes which has been augmented with edges, information sets, actions, players, and utility functions. This paper introduces game morphisms which map source nodes to target nodes in a way which respects this structure.

There is quite a bit of structure. First consider the edges. In this dimension, game morphisms can be defined much like the graph morphisms of the category \mathbf{Grph} (details in note 33). Second consider the information sets. This paper newly regards the collection of information sets as the basis of a topology over the (decision) nodes. So in this dimension, game morphisms can be defined much like the continuous functions of the category \mathbf{Top} (Mac Lane 1998, page 12). Beyond this, three layers of structure remain: actions, players, and utility functions. Much of this paper is dedicated to developing game morphisms which respect these extra layers of structure.

All of this leads to a category for extensive-form games, which is called \mathbf{Gm} . Proposition 3.5 shows that \mathbf{Gm} is well-defined, and Propositions 4.2 and 4.4 characterize its monomorphisms and isomorphisms. These three results suggest that \mathbf{Gm} is coherent, tractable, and meaningful.

Date: March 30, 2026. Essentially identical to Sections 2, 3.1, and 4.1 of Streufert 2021. Now better written, with cleaner definitions of out-tree and player transformation. *Keywords:* Game tree, Continuously labeled tree. *Classifications:* MSC 91A70, JEL C73. *Contact information:* pstreuf@uwo.ca, 519-661-3500, Economics Department, Western University, London, Ontario, N6A 5C2, Canada.

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The remainder of this introduction places **Gm** in the literature, and motivates it in the context of future research.

1.1. PLACEMENT IN LITERATURE

The large literature on game semantics uses specialized games to study logic and computation (some general references are Hyland 1997, Japaridze 2003, and Abramsky and Reggio 2024). This literature borders on the present paper because it often uses category theory to describe the algebra of composing games (see Hyland 1997, page 133). In contrast, the present paper uses category theory to compare games.

In a similar vein, there is a growing computer-science literature which composes extensive-form games from game fragments called “open games”.¹ Within this literature, the papers with nonsingleton information sets are Bolt, Hedges, and Zahn 2023, and Capucci, Ghani, Ledent, and Nordvall Forsberg 2022. These papers define open games as morphisms, which are to be composed to create games. In contrast, the present paper defines games as objects, to be compared by morphisms.

So, closer to the categories here are categories in which the objects are specialized kinds of extensive-form games. Honsell, Lenisa, and Redamalla 2012 builds a category for certain coalgebraic games. Larsson, Nowakowski, and Santos 2018 develops a category for certain extensive-form games on the basis of their plays. Hedges 2018 defines morphisms for some open games (previous paragraph) which have singleton information sets. Duzi, Szeptycki, and Tholen 2024 develops a category which provides a new proof of a topological equivalence. Finally, Bašić, Ellis, Ernst, Popović, and Sieben 2024, and Baltushkin, Ernst, and Sieben 2025 develop categories for certain combinatoric games.²

The author’s research deviates from all the contributions in the previous paragraph by accommodating nonsingleton information sets (i.e., imperfect information). The author’s first approach was restricted to node-and-choice games (i.e., distinguished-action games), which require that each information set has its own actions (see Appendix A.3). Three papers resulted: Streufert 2018 (for “preforms” resembling the CLTs in the present paper), Streufert 2020a (for “forms”), and finally Streufert 2020b (for games). The author’s second and current approach began with Streufert 2021, which removed the distinguished-action restriction. The present paper is a revision which more cleanly defines the concepts of out-tree and player transformation (and for the sake of brevity omits all results on subgames, equilibrium concepts, and subcategories).

¹Incidentally, a different avenue for using categories to construct games is explored in Thomé and Viglizzo 2023.

²Incidentally, Lapitsky 1999 and Jiménez 2014 define categories for simultaneous-move games. Also, Machover and Terrington 2014 defines a category for some cooperative games that are applied in political science. These contributions are not directly relevant since neither simultaneous-move games nor cooperative games are extensive-form games (to be careful, simultaneous-move game can be specified as an extensive-form game in which each player has one information set containing one or more nodes. The extensive-form games considered here are much more general).

1.2. MOTIVATION AND FUTURE RESEARCH

Category theory has added clarity and perspective to many fields of mathematics. So it seems valuable to do the same for the growing field of game theory.³

The author’s own motivation is rooted in the field of economics. Specifically, the extensive-form games used by economists are formulated in many different styles (as demonstrated by the examples of Appendix A). This diversity of style arises not only from historical accident, but also from the different notational needs of different mathematical contexts. Economists informally understand that [a] any “substantial” concept in one style should have the same meaning in any other style, and [b] any “substantial” result in one style should also hold in any other style. This informal understanding might be formally developed. In particular, when a concept or result is translated from one style to another, how would we define the sense in which the translation itself was correct or incorrect? A good answer to this question would help us to identify, manipulate, and communicate the “substance” of game theory. Category theory seems well-suited to this task.

Some progress has already been made. Streufert 2021 shows that [1] the concepts of Nash equilibrium, subgame perfection, perfect information, and no-absentmindedness are each invariant to **Gm** isomorphisms, [2] the full subcategory for distinguished-action games is essentially wide in **Gm**, and [3] the full subcategory for choice-set games is essentially wide in the full subcategory for no-absentminded (i.e., linear) games. The latter two results are discussed by example in Appendix A.4.

The present paper’s class of extensive-form games is broad enough to represent almost all extensive-form games studied by economists (exceptions such as continuous-time extensive-form games are discussed in note 31).⁴ Beyond this are classes of closely related games formulated in very different ways. These include the open games and coalgebraic games mentioned in Section 1.1, and also the pentaform games of Streufert 2025. The author intends to build a category for the latter, and to show its relationship to **Gm**.

1.3. ORGANIZATION

Sections 2 and 3 use two steps to define the category **Gm**. Then Section 4 characterizes monomorphisms and isomorphisms. Appendix A provides examples. Appendices B and C support Sections 2 and 3, respectively. Finally, Appendices D and E support Section 4 regarding monomorphisms and isomorphisms, respectively.

³Some hints of this amorphous value are emerging. [1] This paper’s topological perspective on information sets emerged as the author compared **Gm** with **Top**. That new topological perspective seems to clarify the very definition of a game. [2] Some years ago the author was informed that categorical translations between games may allow for syntactic translations between the logical languages that are interpreted in those games, as in the correspondence theory of van Benthem 2001, and Conradie, Ghilardi, and Palmigiano 2014.

⁴This paper does not formally consider probability. If a game has a finite number of nodes, mixed strategies and expected utilities can be derived by standard means (for example, Myerson 1991, Chapter 4). Meanwhile, if there are an infinite number of nodes, the very concepts of mixed strategy and expected utility can lead to subtle measurability issues.

2. DEFINITION OF **CLT**

2.1. INITIAL REMARKS

It is common to define an extensive-form game in two (and sometimes more) steps. Often the first step is to define an “extensive form”, and the second step is to define an “extensive-form game”. Similarly, this paper’s first step is to define the category **CLT** in Section 2, and its second step is to define the category **Gm** in Section 3. “CLT” stands for “continuously labeled tree” (Definition 2.1). This term is introduced in lieu of “extensive form” because it is more descriptive than “extensive form”, and because there are various meanings of “extensive form” in the literature.

Sections 2 and 3 specify (extensive-form) games as routinely as possible given the literature’s wide variety of game specifications. The particular specification adopted here belongs to the “abstract” style group of Section A.2, resembles the “KS game” of Kline and Luckraz 2016 and Streufert 2019, and is virtually identical to the “traditional game” of Streufert 2025 (details in note 30). In developing this game specification, the author’s guiding principles were to avoid specialized assumptions, to use explicit notation, and to use standard mathematics whenever possible. Nonetheless, almost every reader will find some aspect of this specification unfamiliar, and some of these difficulties are discussed in notes 6, 10, 12, 19, and 20. Excluded kinds of (extensive-form) games are discussed in note 31.

2.2. OUT-TREES

Before defining out-trees, define a *directed graph* to be a couple (X, E) such that $E \subseteq X^2$. The elements of X are called *nodes*. The elements of E are called *edges*, and are denoted by xy rather than $\langle x, y \rangle$. A *walk from w to y in a directed graph (X, E)* is a sequence $\langle x_0, x_1, x_2, \dots, x_m \rangle$ such that

$$(1) \quad x_0 = w, \quad x_m = y, \quad \text{and} \quad (\forall \ell < m) \quad x_\ell x_{\ell+1} \in E.$$

Similarly, an *infinite walk from w in a directed graph (X, E)* is a sequence $\langle x_0, x_1, x_2, \dots \rangle$ such that $x_0 = w$ and $(\forall \ell) \quad x_\ell x_{\ell+1} \in E$. Note that a walk in an arbitrary directed graph can visit a node more than once.

An *out-tree*^{5,6} is a directed graph (X, E) such that

$$(2) \quad (\exists r \in X)(\forall x \in X) \text{ there exists a unique walk from } r \text{ to } x \text{ in } (X, E).$$

⁵Similar definitions in the game-theory literature include Gale and Stewart 1953; Kline and Luckraz 2016, page 88; Streufert 2018; and González-Díaz, García-Jurado, and Fiestras-Janeiro 2023, page 102. Similar concepts in the graph-theory literature include Tutte 1984, page 126; and Bang-Jensen and Gutin 2009, page 21. Several of these definitions apply only to finite trees, and several definitions explicitly include antisymmetry or irreflexivity. The definition here applies equally well to infinite trees, and it implies antisymmetry and thus irreflexivity.

⁶This specification of a tree may be unfamiliar for any of several reasons. First, some game-tree specifications use undirected graphs rather than directed graphs. Second, some specifications use order theory rather than graph theory. Third, some specifications use specialized notations which implicitly impose some of the properties of game trees. For example, some game-tree properties are implicitly imposed if nodes are specified as sequences of past actions, or sets of past actions, or sets of outcomes. Finally, it is common to specify game trees by diagrams rather than by notation.

Out-tree (X, E)		(2) 2.2
X	set of nodes x	2.2
E	set of (directed) edges xy	2.2
r	\hookrightarrow root node	2.2
W	\hookrightarrow set of decision nodes x	(3) 2.2
p	\hookrightarrow immediate predecessor function	2.2
\preceq	\hookrightarrow weak tree order	2.2
\prec	\hookrightarrow strict tree order	2.2
P	\hookrightarrow strict predecessor correspondence	2.2
$X \setminus W$	\hookrightarrow set of end nodes x	2.2
\mathcal{Z}	\hookrightarrow collection of runs Z	(4) 2.2
\mathcal{Z}_{ft}	\hookrightarrow collection of finite runs Z	2.2
$\mathcal{Z}_{\text{inft}}$	\hookrightarrow collection of infinite runs Z	2.2
CLT (Continuously Labeled Tree) $\Theta = (X, E, \mathcal{H}, \lambda)$		(C1)–(C4) 2.3
\mathcal{H}	collection of information sets H	2.3
λ	labeling function (action-assigning function)	2.3
A	\hookrightarrow set of actions a	(5) 2.3
F	\hookrightarrow feasibility correspondence	(7) 2.3
n	\hookrightarrow next-node function	(8) 2.3
CLT morphism $\theta = [\Theta, \Theta', \tau]$		(cE),(cH),(cL) 2.4
τ	node transformation	2.4
α	\hookrightarrow action transformation	(11) 2.4
ζ	\hookrightarrow run transformation (requires θ is end-preserving)	(19) 3.2
Game $\Gamma = (X, E, \mathcal{H}, \lambda, \kappa, U)$		(G1)–(G3) 3.1
κ	control-assigning function (player-assigning function)	3.1
I	\hookrightarrow set of players i	(15) 3.1
U	profile $\langle U_i \rangle_{i \in I}$ listing utility function U_i for each player i	3.1
\lesssim_i	\hookrightarrow total preorder of \mathcal{Z} implied by U_i	(17) 3.1
Game morphism $\gamma = [\Gamma, \Gamma', \tau]$		(gZ),(gK),(gU) 3.2
ι	\hookrightarrow player transformation	(21) 3.2

TABLE 2.1. Terms defined in Sections 2 and 3. Out-trees, CLTs, CLT morphisms, games, and game morphisms are implicitly accompanied by their components and derivatives (\hookrightarrow). Definitions are in equations and sections on right side.

(The term “out-tree” is used instead of its synonyms “outwardly directed rooted tree” and “diverging arborescence” only because it is shorter.) It can be shown that each out-tree has exactly one r that satisfies (2). Call this the *root node* of the out-tree. For example, $(\{5\}, \emptyset)$ is a one-node out-tree because the one-node sequence $\langle 5 \rangle$ is the unique walk from 5 to itself in $(\{5\}, \emptyset)$. In this example, 5 is the root node r . Such one-node out-trees are the “smallest” of out-trees because (\emptyset, \emptyset) is not an out-tree (it lacks the node r required by (2)). In the opposite direction, the “largest” of out-trees

can have uncountably infinite degree (note 8) and countably infinite height (there can be infinite walks of the sort defined below (1)).

Consider an out-tree (X, E) . Let

$$(3) \quad W = \pi_1 E,$$

where $\pi_1 E$ is the projection of E onto its first coordinate. Call W the *decision-node* set. It can be shown that $E^{-1} = \{ \langle y, x \rangle \mid xy \in E \}$ is the graph of a surjective function⁷ from the set $X \setminus \{r\}$ of nonroot nodes to the decision-node set W . Let p denote this function, and call it the *immediate-predecessor function*.⁸ (If the out-tree (X, E) is of the extreme form $(\{x\}, \emptyset)$, then W , $X \setminus \{r\}$, and p are all empty.)

Further, let \preceq be the binary relation on X defined by $x \preceq y$ iff there is a walk from x to y . Call \preceq the *weak tree order*. Note \preceq is reflexive because of the one-node walk from any node to itself. Next, let \prec be the binary relation on X defined by $x \prec y$ iff $x \preceq y$ and $x \neq y$. Call \prec the *strict tree order*. It can be shown that \preceq is a partial order on X and that \prec is its asymmetric part. Further, define the correspondence (note 7) $P: X \rightrightarrows W$ by $P(y) = \{ x \in X \mid x \prec y \}$. Call P the *strict predecessor correspondence*. It is easily shown that $(\forall y \in X \setminus \{r\}) p(y) \in P(y)$.

It can be shown that each walk in the out-tree (X, E) visits each of its nodes exactly once,⁹ and that each walk in (X, E) is characterized by the set of nodes that it visits. Because of this, each walk in (X, E) will be identified with the set of nodes that it visits. For example, for each node $y \in X$, the set $P(y) \cup \{y\}$ is the walk from the root node r to the node y (expressed as a set rather than a sequence).

Further, call $X \setminus W$ the set of *end nodes*, and let \mathcal{Z}_{ft} be the collection of all walks from r to an end node (each walk is expressed as a set $Z \subseteq X$). Further, let \mathcal{Z}_{inf} be the collection of all infinite walks from r (as defined below (1), but each walk is expressed as a set $Z \subseteq X$). Finally, let

$$(4) \quad \mathcal{Z} = \mathcal{Z}_{\text{ft}} \cup \mathcal{Z}_{\text{inf}},$$

and call \mathcal{Z} the collection of *runs* (elsewhere “plays”). Note that \mathcal{Z} is nonempty (if the out-tree (X, E) is of the extreme form $(\{x\}, \emptyset)$, then the end-node set is $X \setminus W = \{x\}$ and the only run in \mathcal{Z} is $Z = \{x\}$). Possibly $\mathcal{Z} = \mathcal{Z}_{\text{ft}}$, possibly $\mathcal{Z} = \mathcal{Z}_{\text{inf}}$, and possibly both \mathcal{Z}_{ft} and \mathcal{Z}_{inf} are nonempty.

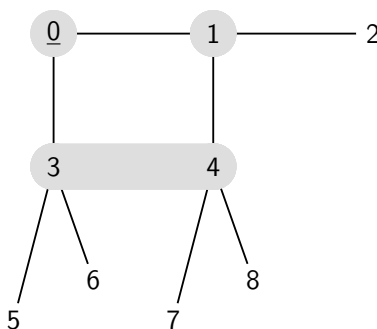
2.3. CONTINUOUSLY LABELED TREES (CLTs)

Consider an out-tree (X, E) . Let \mathcal{H} be a partition of the decision-node set W , and call its elements *information sets*. For example, consider Figure 2.1. There

⁷To be clear, a *correspondence* $F: X \rightrightarrows Y$ is taken to be a triple $[X, Y, F^{\text{gr}}]$ such that $F^{\text{gr}} \subseteq X \times Y$, and a *function* $f: X \rightarrow Y$ is taken to be a triple $[X, Y, f^{\text{gr}}]$ such that $f^{\text{gr}} \subseteq X \times Y$ and $(\forall x \in X)(\exists! y \in Y)(x, y) \in f^{\text{gr}}$ (the three components of a correspondence or function are its domain, codomain, and graph; the term “correspondence” is standard in economics but perhaps not elsewhere).

⁸Incidentally, the inverse image of p at x is $\{y \mid x = p(y)\} = \{y \mid xy \in E\}$. This could be called the set of x ’s “immediate successors”. A node can have uncountably many immediate successors. In other words, an out-tree can have uncountable degree.

⁹This implies that each walk in (X, E) is a “path”, where a “path” is a special kind of walk which visits each of its nodes exactly once. This paper does not use the term “path”.

FIGURE 2.1. An out-tree (X, E) with a partition \mathcal{H} of its decision-node set W

the decision-node set is $W = \{0, 1, 3, 4\}$, and the shaded regions depict that W is partitioned by the information sets belonging to the collection $\mathcal{H} = \{\{0\}, \{1\}, \{3, 4\}\}$.

One can regard \mathcal{H} as the basis of a topology for W . Call this topology the *information topology* for W . Thus a set of decision nodes is open in the information topology for W iff it is the union of a collection of information sets from \mathcal{H} . This paper uses the information topology for W and the discrete topology everywhere else.

Next, let λ be a surjective function from the edge set E to a discrete (i.e. unstructured) codomain. Call λ the *labeling function* (elsewhere an “action-assigning function”),¹⁰ and say that $\lambda(xy)$ labels the edge xy . Next define A to be the codomain of λ . In other words (since λ is surjective), let¹¹

$$(5) \quad A = \bar{\lambda}(E),$$

where $\bar{\lambda}(E) = \{ \lambda(xy) \mid xy \in E \}$ denotes the image of the set E . Although A could be called the set of “labels”, A will instead be called the set of *actions*.

It will be assumed that λ is *locally injective* in the sense that, for any two edges of the form $xy_1 \in E$ and $xy_2 \in E$,

$$(6) \quad y_1 \neq y_2 \text{ implies } \lambda(xy_1) \neq \lambda(xy_2).$$

This implies that the edges leaving each decision node $x \in W$ are injectively labeled. For example, the labeling function λ shown in Figure 2.2 is locally injective even

¹⁰This specification of a labeling function may be unfamiliar for any of several reasons. First, there is a bijection between the edge set E and the successor-node set $X \setminus \{r\}$. Because of this, a labeling function λ on E could be alternatively specified by a function on $X \setminus \{r\}$. Second, some specifications use specialized notations which implicitly assign actions to edges or successor nodes. For example, if nodes are specified as sequences of past actions, then the action assigned to a successor node is simply the node’s most recent past action. Third, a locally injective labeling function λ can be expressed as a map from decision-node/feasible-action couples to nodes, as suggested by the map n in note 14. Fourth, it is common to assign actions to edges in a tree diagram rather than through formal notation.

¹¹To be clear, recall from note 7 that a function is regarded as a triple. Here, $\lambda = [\lambda^{\text{dom}}, \lambda^{\text{cod}}, \lambda^{\text{gr}}]$ is assumed to be a surjective function from E . This implies $\lambda^{\text{dom}} = E$ and $\lambda^{\text{cod}} = \bar{\lambda}(E)$. Next, A is defined to be $\lambda^{\text{cod}} = \bar{\lambda}(E)$. Since A is derived from λ , A will not appear in the tuple defining a CLT. Accordingly, Table 2.1 lists λ as a component of a CLT, and A as a derivative of a CLT.

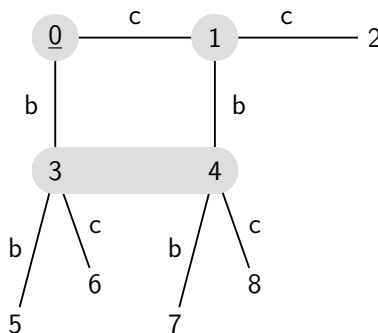


FIGURE 2.2. A continuously labeled tree (CLT). A CLT consists of an out-tree (X, E) , a partition \mathcal{H} , and a labeling (action-assigning) function λ .

though it is not injective. Local injectivity is equivalent to the concept of “determinism” in Blackburn, de Rijke, and Venema 2001, page 3.

From λ derive the correspondence $F:W \rightrightarrows A$ by

$$(7) \quad (\forall x \in W) F(x) = \{a \in A \mid (\exists y) \lambda(xy) = a\}.$$

Thus $F(x)$ is the set of actions that label the edges leaving x . Call F the *feasibility correspondence*, and call $F(x)$ the set of actions that are *feasible* at x .¹² For example, $F(3) = \{b, c\}$ in Figure 2.2.

For future reference, we define a function n which takes each decision node $x \in W$ and each feasible action $a \in F(x)$ to the node $n(x, a)$ at the end of the edge from x labeled by a . Casually, $n(x, a)$ is the result of choosing a at x . Formally, assume λ is locally injective and define¹³ $n: F^{\text{gr}} \rightarrow X \setminus \{r\}$ implicitly by¹⁴

$$(8) \quad (\forall (x, a) \in F^{\text{gr}}) \lambda(x n(x, a)) = a.$$

Thus $n(x, a)$ is defined to be the unique y_* such that $\lambda(xy_*) = a$. Call n the *next-node function* (it is well-defined by Lemma B.1(b)). For example, $n(3, c) = 6$ in Figure 2.2.

From a different perspective, $F = \langle F(x) \rangle_{x \in W}$ can be regarded as a set-valued function from W to $\mathcal{P}(A)$. It will be assumed that F is continuous as a function from W , endowed with the information topology, to $\mathcal{P}(A)$, endowed with the discrete topology.¹⁵ This is equivalent to

$$(9) \quad (\forall H \in \mathcal{H}, x_1 \in H, x_2 \in H) F(x_1) = F(x_2),$$

¹²This derivation of the feasibility correspondence may be unfamiliar because the literature occasionally has the feasible set as a function of the information set rather than the decision node. In this alternative, a well-defined feasibility correspondence implicitly imposes continuity (9).

¹³ F^{gr} is the graph of the correspondence F , as in note 7. Notice that $(x, a) \in F^{\text{gr}}$ iff $a \in F(x)$.

¹⁴Casually, $\lambda(x \cdot)$ and $n(x, \cdot)$ are local inverses at each decision node x . More specifically, $\lambda(x \cdot)$ maps immediate successors y to feasible actions a , and conversely, $n(x, \cdot)$ maps feasible actions a to immediate successors y . Details are in Lemma B.1(c).

¹⁵This is equivalent to assuming that $F:W \rightrightarrows A$ is lower semicontinuous in the sense of Berge 1963, page 109, when its domain is endowed with the information topology, and when its codomain is endowed with the discrete topology.

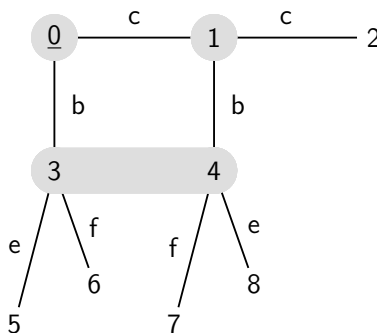


FIGURE 2.3. A CLT which has different actions than Figure 2.2

which requires that two nodes in one information set have the same feasible set. For example, nodes 3 and 4 in Figure 2.3 share an information set, and in accord with (9), the nodes share the feasible set $F(3) = F(4) = \{e, f\}$.

Definition 2.1. A continuously labeled tree (CLT) is a tuple $\Theta = (X, E, \mathcal{H}, \lambda)$ such that

- (C1) (X, E) is an out-tree (2),
- (C2) \mathcal{H} is a partition of W ,
- (C3) λ is a locally injective (6) surjective function from E , and
- (C4) $\langle F(x) \rangle_{x \in W}$ is continuous (9) from W

(where E determines W by (3) and λ determines F by (7)).

Note that a tree diagram with underlined root node, shaded information sets, and labeled edges unambiguously specifies all the components of a CLT. Thus each of Figures 2.2 and 2.3 defines a CLT. In a different direction, note that the “smallest” CLTs have a one-node out-tree $(X, E) = (\{x\}, \emptyset)$, an empty partition \mathcal{H} , and an empty labeling function λ . Here the decision-node set W is empty.

2.4. CLT MORPHISMS

This Section 2.4 will define CLT morphisms. This first paragraph fixes some general notation. For $f: X \rightarrow Y$ and $A \subseteq X$, let $\bar{f}(A) = \{f(x) \mid x \in A\}$. Also, for $f: X \rightarrow Y$, $A \subseteq X$, and $B \subseteq Y$, let $f|_{A,B}$ be the function with domain A , codomain B , and the graph of $f|_A$. This construction is well-defined iff $\bar{f}(A) \subseteq B$.

Consider a source CLT Θ and a target CLT Θ' . A *node transformation* is a function of the form $\tau: X \rightarrow X'$. Definition 2.4 will assume that τ *preserves edges* in the sense that

$$(10) \quad (\forall xy \in E) \tau(x)\tau(y) \in E'.$$

Edge preservation implies that $[(X, E), (X', E'), \tau]$ is a graph morphism, and easily leads to the several consequences in Proposition 2.5(a–e). Among these consequences is $\bar{\tau}(W) \subseteq W'$, which implies that $\tau|_{W,W'}$ is well-defined (recall the general notation of the previous paragraph).

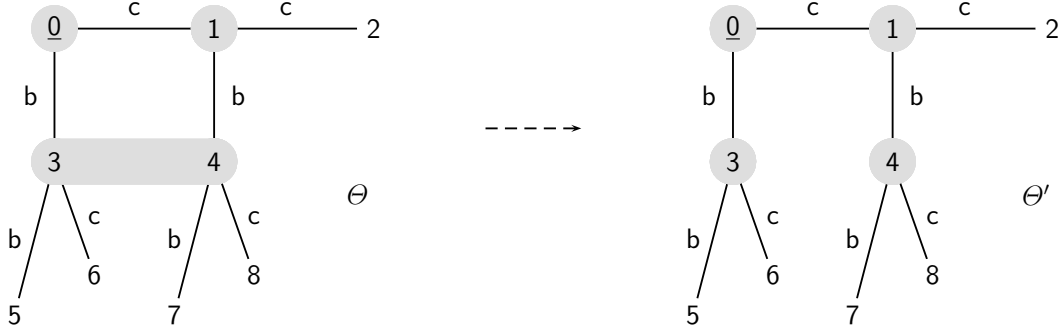


FIGURE 2.4. The CLT Θ (from Figure 2.2) and the CLT Θ' have different information sets. The tuple $[\Theta, \Theta', \text{id}_X]$ is not a morphism because $\tau|_{W, W'} = \text{id}_W$ is not continuous.

Definition 2.4 will also assume that $\tau|_{W, W'}$ is continuous. As would be expected from general topology, this means that the inverse image of every target information set is the union of a collection of source information sets. However, Proposition 2.2 provides another perspective by taking advantage of the fact that W and W' are endowed with partition topologies. The proposition shows that continuity is equivalent to the image of each source information set being included in a target information set. In other words, continuity says that source information sets cannot be “split”.

Proposition 2.2. *Suppose Θ and Θ' are CLTs, $\tau: X \rightarrow X'$, and $\bar{\tau}(W) \subseteq W'$. Then $\tau|_{W, W'}$ is continuous iff $(\forall H \in \mathcal{H})(\exists H' \in \mathcal{H}') \bar{\tau}(H) \subseteq H'$. (Proof B.2.)*

For example, consider Figure 2.4. There $\tau|_{W, W'} = \text{id}_W$ is discontinuous because the inverse image $\{x \in W \mid \tau(x) \in \{4\}\} = \{4\}$ of the target information set $\{4\}$ fails to be open in the source information topology. Equivalently, $\tau|_{W, W'} = \text{id}_W$ is discontinuous because the image $\bar{\tau}(\{3, 4\}) = \{3, 4\}$ of the source information set $\{3, 4\}$ is not included in a target information set. In other words, $\tau|_{W, W'} = \text{id}_W$ is discontinuous because it “splits” the source information set $\{3, 4\}$. In contrast, consider Figure 2.5 or Figure 2.6 or Figure 4.4. In each of those cases, $\tau|_{W, W'} = \text{id}_W$ is continuous since no source information sets are split.

To interpret continuity in the context of game theory, we only need two observations: [1] that large information sets correspond to less information, and [2] that continuity prevents information sets from being split. Thus, continuity corresponds to preserving a lack of information.

Next, from the tuple $[\Theta, \Theta', \tau]$, derive $\alpha = \langle \alpha_x: F(x) \rightarrow F'(\tau(x)) \rangle_{x \in W}$ at each decision node $x \in W$ by

$$(11) \quad (\forall a \in F(x)) \alpha_x(a) = \lambda'(\tau(x) \tau(n(x, a)))$$

(where the source labeling function λ determines the source next-node function n by (8)). Call $\alpha = \langle \alpha_x \rangle_{x \in W}$ the tuple’s *action transformation*. At each $x \in W$, the statement $\alpha_x: F(x) \rightarrow F'(\tau(x))$ means that the function α_x sends source actions feasible at x to target actions feasible at the image of x . In particular, consider a source action a which is feasible at x . By (11), this a is sent to the target action that labels the

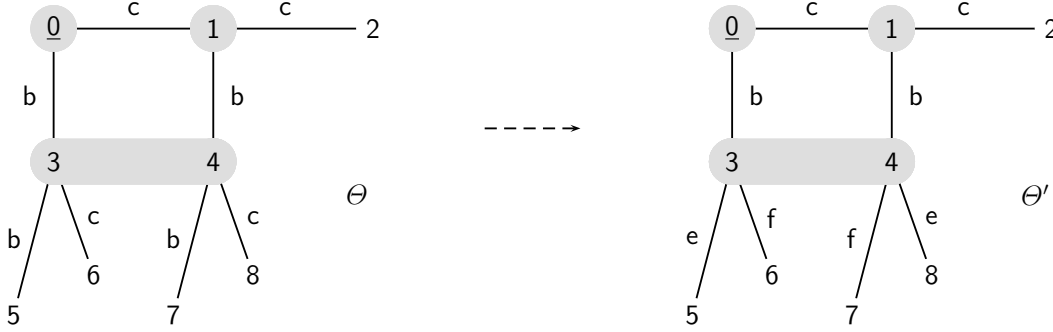


FIGURE 2.5. The CLT Θ (from Figure 2.2) and the CLT Θ' (from Figure 2.3) have different actions. The tuple $[\Theta, \Theta', \text{id}_X]$ is not a morphism because its $\alpha = \langle \alpha_x \rangle_{x \in W}$ is not continuous (12) from W .

target edge from the image of x to the image of the next source node determined by x and a .

Proposition 2.3. *Suppose Θ and Θ' are CLTs, and $\tau: X \rightarrow X'$ preserves edges (10). Then the action transformation $\alpha = \langle \alpha_x \rangle_{x \in W}$ (11) is well-defined. (Proof B.3.)*

As a whole, $\alpha = \langle \alpha_x \rangle_{x \in W}$ is a function-valued function with domain W . Definition 2.4 will assume that α is continuous as a function from W , endowed with the information topology, to the set of functions, endowed with the discrete topology. This is equivalent to assuming that

$$(12) \quad (\forall H \in \mathcal{H}, x_1 \in H, x_2 \in H) \alpha_{x_1} = \alpha_{x_2}.$$

It can be shown¹⁶ that if $\tau|_{W, W'}$ is continuous, then (12) is equivalent to

$$(13) \quad (\forall H \in \mathcal{H}, x_1 \in H, x_2 \in H) (a \in F(x_1)) \alpha_{x_1}(a) = \alpha_{x_2}(a).$$

In this precise sense, the continuity of α is equivalent to requiring that two nodes in the same information set transform actions in the same way.

For example, consider Figure 2.5. Here

$$\alpha_3(\mathbf{b}) = \lambda'(\tau(3) \tau(n(3, \mathbf{b}))) = \lambda'(\tau(3) \tau(5)) = \lambda'(35) = \mathbf{e} \text{ and}$$

$$\alpha_4(\mathbf{b}) = \lambda'(\tau(4) \tau(n(4, \mathbf{b}))) = \lambda'(\tau(4) \tau(7)) = \lambda'(47) = \mathbf{f}.$$

Since these differ, $\alpha_3 \neq \alpha_4$. Thus (12) fails since 3 and 4 belong to the same source information set $H = \{3, 4\} \subseteq W$. Hence α is discontinuous from W . In contrast, Figure 2.6 provides an example of a continuous α . There $\alpha_3(\mathbf{b}) = \alpha_4(\mathbf{b}) = \mathbf{e}$ and $\alpha_3(\mathbf{c}) = \alpha_4(\mathbf{c}) = \mathbf{f}$. Thus by (13), $\alpha_3 = \alpha_4$. So by (12) and the fact that $H = \{3, 4\} \subseteq W$ is the only nonsingleton source information set, α is continuous from W .

¹⁶Take a source information set $H \in \mathcal{H}$ and two decision nodes x_1 and x_2 in H . It suffices to show that the domains and codomains of $\alpha_{x_1}: F(x_1) \rightarrow F'(\tau(x_1))$ and $\alpha_{x_2}: F(x_2) \rightarrow F'(\tau(x_2))$ are equal. Axiom (C4) for Θ implies $F(x_1) = F(x_2)$. Further, Lemma 2.2 and the assumed continuity of $\tau|_{W, W'}$ implies there is $H' \in \mathcal{H}'$ such that $\{\tau(x_1), \tau(x_2)\} \subseteq H'$. Thus (C4) for Θ' implies $F'(\tau(x_1)) = F'(\tau(x_2))$.

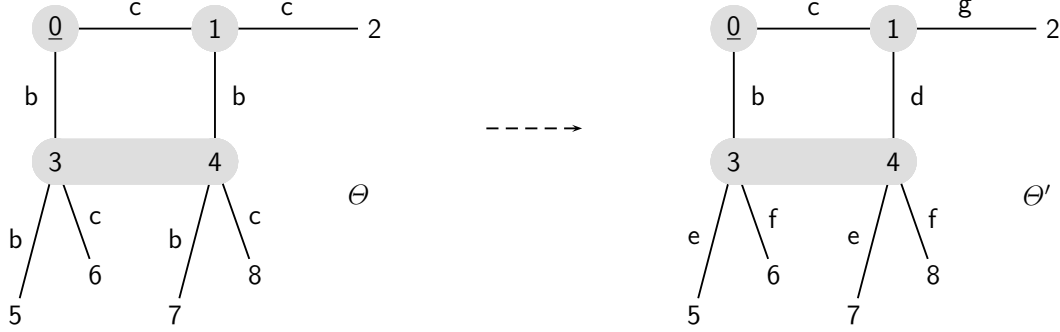


FIGURE 2.6. The CLT Θ (from Figure 2.2) and the CLT Θ' have different actions. The tuple $[\Theta, \Theta', \text{id}_X]$ is a morphism.

Note that the continuity of α from W allows “local” action changes in the sense that actions can be changed differently at different information sets. For example, consider Figure 2.6, and note $\alpha_0(\mathbf{b}) = \mathbf{b}$ and $\alpha_3(\mathbf{b}) = \mathbf{e}$. This is consistent with the continuity of α because 0 and 3 are in different information sets.

Definition 2.4. A **CLT morphism** is a tuple $\theta = [\Theta, \Theta', \tau]$ such that Θ and Θ' are CLTs (Definition 2.1), $\tau: X \rightarrow X'$,

(cE) τ is edge-preserving (10),

(cH) $\tau|_{W, W'}$ is continuous, and

(cL) $\langle \alpha_x \rangle_{x \in W}$ is continuous (12) from W

(where $[\Theta, \Theta', \tau]$ determines $\langle \alpha_x \rangle_{x \in W}$ by (11)).

Mnemonically, (cE) abbreviates “edge preservation”, (cH) “information-set preservation”, and (cL) “label preservation” (the leading “c” in (cE), (cH), and (cL) signifies “CLT”). Note that the source root r may or may not be taken to the target root r' .

Proposition 2.5. Suppose $[\Theta, \Theta', \tau]$ is a morphism. Then the following hold.

(a) $(\forall x \in X, y \in X) x \preceq y \Rightarrow \tau(x) \preceq' \tau(y)$.

(b) $(\forall x \in X, y \in X) x \prec y \Rightarrow \tau(x) \prec' \tau(y)$.

(c) $(\forall y \in X) \bar{\tau}(P(y)) \subseteq P'(\tau(y))$.¹⁷

(d) $(\forall y \in X \setminus \{r\}) \tau(p(y)) = p'(\tau(y))$.

(e) $\bar{\tau}(W) \subseteq W'$.¹⁷

(f) $(\forall xy \in E) \alpha_x(\lambda(xy)) = \lambda'(\tau(x)\tau(y))$.

(g) $(\forall x \in W, a \in F(x)) \tau(n(x, a)) = n'(\tau(x), \alpha_x(a))$. (Proof B.4.)

2.5. THE CATEGORY CLT

This paragraph defines the category **CLT**, which is called the *category of CLTs* (continuously labeled trees). Let an object be a CLT Θ (Definition 2.1). Let an

¹⁷Recall $\bar{f}(A)$ was defined as $\{f(x) \mid x \in A\}$ shortly after the definition of a CLT.

arrow be a CLT morphism $[\Theta, \Theta', \tau]$ (Definition 2.4). Let source, target, identity, and composition be

$$(14) \quad \begin{aligned} [\Theta, \Theta', \tau]^{\text{src}} &= \Theta, \quad [\Theta, \Theta', \tau]^{\text{trg}} = \Theta', \quad \text{id}_{\Theta} = [\Theta, \Theta, \text{id}_X], \quad \text{and} \\ [\Theta', \Theta'', \tau'] \circ [\Theta, \Theta', \tau] &= [\Theta, \Theta'', \tau' \circ \tau] \end{aligned}$$

(where id_X is an identity in \mathbf{Set}).

Proposition 2.6.

- (a) **CLT** is well-defined.
- (b) If Θ is a CLT, then the action transformation of id_{Θ} is $\langle \text{id}_{F(x)} \rangle_{x \in W}$.
- (c) If $[\Theta, \Theta', \tau]$ and $[\Theta', \Theta'', \tau']$ are morphisms, then the action transformation of $[\Theta', \Theta'', \tau'] \circ [\Theta, \Theta', \tau]$ is $\langle \alpha'_{\tau(x)} \circ \alpha_x \rangle_{x \in W}$. (Proof B.7.)

3. DEFINITION OF \mathbf{Gm}

3.1. GAMES

To define a game, begin with a CLT $\Theta = (X, E, \mathcal{H}, \lambda)$. Then let κ be a surjective function from the decision-node set W to a discrete (i.e. unstructured) codomain. Define I to be the codomain of κ .¹⁸ In other words (since κ is surjective), let

$$(15) \quad I = \bar{\kappa}(W).$$

Call κ the *control-assigning function* (elsewhere the “player-assigning function” or “move-assigning function”),¹⁹ call a member of I a *player*, and say that player $\kappa(x)$ *controls* node x .

Definition 3.1 will assume that κ is continuous. This means that, for each $i \in I$, the inverse image $\{x \in W \mid \kappa(x) = i\}$ is open in the information topology. This is equivalent to saying that, for each $i \in I$, $\{x \in W \mid \kappa(x) = i\}$ is the union of a collection of information sets. Equivalently,

$$(16) \quad (\forall H \in \mathcal{H}, x_1 \in H, x_2 \in H) \quad \kappa(x_1) = \kappa(x_2),$$

which requires that two nodes in one information set are assigned to the same player.

Next, for each $i \in I$, consider a function $U_i: \mathcal{Z} \rightarrow \bar{\mathbb{R}}$, where $\bar{\mathbb{R}}$ denotes the set of extended real numbers, that is, $\mathbb{R} \cup \{-\infty, \infty\}$.²⁰ Call U_i the *utility function* of player i

¹⁸This derivation of I from κ is very similar to the derivation of A from λ . In particular, note 11 would apply with I, κ, W , and “game” replacing, respectively, A, λ, E , and “CLT”.

¹⁹This specification of a control-assigning function may be slightly unfamiliar. First, the term “control-assigning” is introduced here since it is less ambiguous than “player-assigning” or “move-assigning”. Second, some other specifications partition the decision-node set, and then label the partition’s cells by players. This is effectively a map from decision nodes to players. Third, a few other specifications map information sets to players. This implies a map from decision nodes to players and implicitly imposes continuity (16).

²⁰This specification of a utility function may be slightly unfamiliar. First, in finite-horizon games, it is common to assign utility numbers to end nodes rather than to runs. In this special case, there is a bijection between the end-node set $X \setminus W$ and the run collection \mathcal{Z} (here \mathcal{Z} equals the collection \mathcal{Z}_{ft} of finite runs, as defined prior to (4)). Second, infinite utility numbers are included because, in economics, many popular utility functions generate $-\infty$ utility when some consumption level is

(elsewhere the “payoff function” of player i). From the function $U_i: \mathcal{Z} \rightarrow \bar{\mathbb{R}}$, derive the total preorder²¹ \lesssim_i on \mathcal{Z} defined by

$$(17) \quad (\forall Z_1 \in \mathcal{Z}, Z_2 \in \mathcal{Z}) \quad Z_1 \lesssim_i Z_2 \text{ iff } U_i(Z_1) \leq U_i(Z_2).$$

In economic theory, the function U_i is said to “represent” the total preorder \lesssim_i .

Definition 3.1. *An (extensive-form) game is a tuple $\Gamma = (X, E, \mathcal{H}, \lambda, \kappa, U)$ such that*

$$(G1) \quad (X, E, \mathcal{H}, \lambda) \text{ is a CLT (Definition 2.1),}$$

$$(G2) \quad \kappa \text{ is a continuous (16) surjective function from } W, \text{ and}$$

$$(G3) \quad U \text{ is of the form } \langle U_i: \mathcal{Z} \rightarrow \bar{\mathbb{R}} \rangle_{i \in I}$$

(where (X, E) determines W and \mathcal{Z} by (3) and (4), and I is κ 's codomain (15)).

Figure A.1 in Appendix A provides six examples. Note that a CLT diagram with controlling players and utility vectors unambiguously specifies almost all the components of a game. The only thing that may need to be specified externally is the order in which the utility vectors list the players. In a different direction, note that the “smallest” games have one-node out-tree $(\{x\}, \emptyset)$, an empty partition \mathcal{H} , an empty labeling function λ , an empty control-assigning function κ , and an empty utility-function vector U . Here the player set I is empty.

3.2. GAME MORPHISMS

Game morphisms will be built on certain CLT morphisms. Say that a CLT morphism $[\Theta, \Theta', \tau]$ *preserves ends* iff

$$(18) \quad \bar{\tau}(X \setminus W) \subseteq X' \setminus W'.$$

Figure 3.1 shows a morphism which does not preserve ends, because the end node 2 is mapped to the decision node 12. Accordingly, the run $\{0, 2\}$ is mapped to the non-run walk $\{10, 12\}$. Arguably, the source utility assigned to the source run $\{0, 2\}$ should imply nothing about the target utilities assigned to the target runs $\{10, 12, 13\}$ and $\{10, 12, 14\}$ because the connection between that source run and those two target runs is so nebulous. For this reason, game morphisms are not built on CLT morphisms that fail to preserve ends.

In contrast, Figure 3.2 shows an end-preserving morphism. Here it seems reasonable that a player's preference between the source runs $\{0, 1\}$ and $\{0, 2\}$ could be preserved in a preference between the target runs $\{50, 10, 11\}$ and $\{50, 10, 12\}$. Note that the images $\{10, 11\}$ and $\{10, 12\}$ of the two source runs are preceded by the target node 50, and that 50 is not the image of any source node.

zero. Such a utility function is often part of a consumer dynamic optimization problem, and such a problem can be specified as a one-player game. Third, preferences are occasionally specified by a total preorder like (17) rather than by a utility function.

²¹To be clear, a total preorder is a total and transitive binary relation. In economic theory, totality is called “completeness” and a total preorder is called an “ordering”.

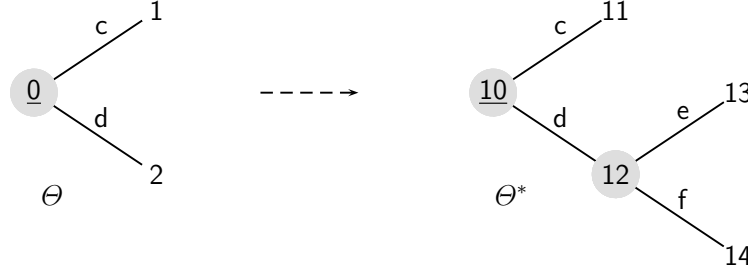


FIGURE 3.1. The CLT morphism $[\Theta, \Theta^*, \tau]$ defined by $\tau(x)=x+10$. This morphism is not end-preserving (18).

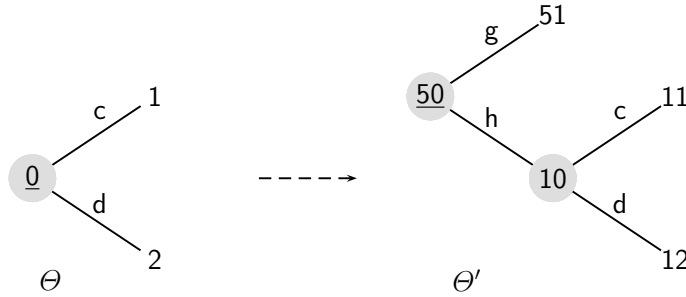


FIGURE 3.2. The CLT morphism $[\Theta, \Theta', \tau]$ defined by $\tau(x)=x+10$. This morphism is end-preserving (18).

From an end-preserving $[\Theta, \Theta', \tau]$, derive the function $\zeta: \mathcal{Z} \rightarrow \mathcal{Z}'$ by

$$(19) \quad \zeta(Z) = P' \circ \tau(r) \cup \bar{\tau}(Z).$$

Call ζ the *run transformation*. Proposition 3.2 shows that ζ is well-defined. For example, in Figure 3.2,

$$\zeta(\{0, 2\}) = P' \circ \tau(r) \cup \bar{\tau}(\{0, 2\}) = \{50\} \cup \{10, 12\} = \{50, 10, 12\},$$

where $P' \circ \tau(r) = P' \circ \tau(0) = P'(10) = \{50\}$.

Proposition 3.2. *If a morphism $[\Theta, \Theta', \tau]$ is end-preserving (18), then its run transformation ζ (19) is well-defined. (Proof C.1.)*

Next, consider two games Γ and Γ' together with a run transformation $\tau: X \rightarrow X'$. The triple $[\Gamma, \Gamma', \tau]$ is said to *preserve control* iff²²

$$(20) \quad (\forall x_1 \in W, x_2 \in W) \kappa(x_1) = \kappa(x_2) \text{ implies } \kappa'(\tau(x_1)) = \kappa'(\tau(x_2)).$$

This requires that any two source nodes controlled by one player must be sent to target nodes controlled by one player. Figure 3.3 shows an example which violates this assumption.

From a control-preserving tuple $[\Gamma, \Gamma', \tau]$, derive the function $\iota: I \rightarrow I'$ with graph

$$(21) \quad \iota^{\text{gr}} = \{ (\kappa(x), \kappa'(\tau(x))) \mid x \in W \}.$$

²²In (20)'s concluding equation, each side is well-defined because $(\forall w \in W) \tau(w) \in W'$ by Proposition 2.5(e).

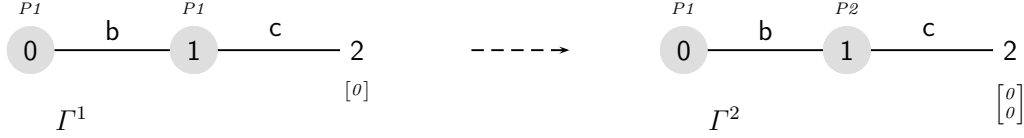


FIGURE 3.3. The games Γ^1 and Γ^2 have different players. The tuple $[\Gamma^1, \Gamma^2, \text{id}_{X^1}]$ is not a game morphism since it is not control-preserving (20).

Call ι the *player transformation* of $[\Gamma, \Gamma', \tau]$. Equation (21) says that ι maps the player controlling decision node x to the player controlling the image of x . Proposition 3.3(a) shows that ι is well-defined. Further, Proposition 3.3(b) implies that $(\forall x \in W) \iota(\kappa(x)) = \kappa'(\tau(x))$, which says that for each decision node $x \in W$, the source player controlling x is sent by ι to the target player controlling the image of x .

Proposition 3.3. *Suppose that Γ and Γ' are games, that $\tau: X \rightarrow X'$, and that $[\Gamma, \Gamma', \tau]$ is a control-preserving (20) tuple. Then (a) the player transformation ι (21) is well-defined, and (b) $\iota \circ \kappa = \kappa' \circ \tau|_{W, W'}$. (Proof C.2.)*

Definition 3.4. *A game morphism is a tuple $\gamma = [\Gamma, \Gamma', \tau]$ such that Γ and Γ' are games,*

(gZ) $[\Theta, \Theta', \tau]$ is an end-preserving (18) CLT morphism (Definition 2.4),

(gK) $[\Gamma, \Gamma', \tau]$ is control-preserving (20), and

(gU) $(\forall i \in I) \zeta$ is monotone from $(\mathcal{Z}, \lesssim_i)$ to $(\mathcal{Z}', \lesssim'_{\iota(i)})$

(where $[\Gamma, \Gamma', \tau]$ determines ζ and ι by (19) and (21)).

To summarize briefly, (gZ) explicitly preserves ends, and implicitly preserves edges, information sets and labels via the Definition 2.4 of a CLT morphism. Then (gK) preserves control. Finally, the definition (17) of \lesssim_i implies that (gU) is equivalent to

$$(22) \quad (\forall i \in I, Z_1 \in \mathcal{Z}, Z_2 \in \mathcal{Z}) U_i(Z_1) \leq U_i(Z_2) \text{ implies } U_{\iota(i)}(\zeta(Z_1)) \leq U_{\iota(i)}(\zeta(Z_2)).$$

In this sense, (gU) preserves the ordinal content of the utility functions.

3.3. THE CATEGORY **Gm**

This paragraph defines the category **Gm**, which is called the *category of extensive-form games*. Let an object be a game Γ (Definition 3.1). Let an arrow be a game morphism $[\Gamma, \Gamma', \tau]$ (Definition 3.4). Let source, target, identity, and composition be

$$(23) \quad [\Gamma, \Gamma', \tau]^{\text{src}} = \Gamma, [\Gamma, \Gamma', \tau]^{\text{trg}} = \Gamma', \text{id}_\Gamma = [\Gamma, \Gamma, \text{id}_X], \text{ and} \\ [\Gamma', \Gamma'', \tau'] \circ [\Gamma, \Gamma', \tau] = [\Gamma, \Gamma'', \tau' \circ \tau].$$

Proposition 3.5.

(a) **Gm** is well-defined.

(b) Suppose Γ is a game. Then the run transformation of id_Γ is $\text{id}_\mathcal{Z}$, and its player transformation is id_I .

(c) Suppose $[\Gamma, \Gamma', \tau]$ and $[\Gamma', \Gamma'', \tau']$ are morphisms. Then the run transformation of $[\Gamma', \Gamma'', \tau'] \circ [\Gamma, \Gamma', \tau]$ is $\zeta' \circ \zeta$, and its player transformation is $\iota' \circ \iota$. (Proof C.6.)

Proposition 3.6. Define F from \mathbf{Gm} to \mathbf{CLT} by

$$F_0 : (X, E, \mathcal{H}, \lambda, \kappa, U) \mapsto (X, E, \mathcal{H}, \lambda) \text{ and}$$

$$F_1 : [\Gamma, \Gamma', \tau] \mapsto [F_0(\Gamma), F_0(\Gamma'), \tau].$$

Then F is a well-defined functor. (Proof C.7.)

The “forgetful” functor of Proposition 3.6 allows Section 2’s results for CLTs to be applied to games in a natural but rigorous way. For example, Propositions 3.6 and 2.6(b) imply that if Γ is a game, then the action transformation of id_Γ is $\langle \text{id}_{F(x)} \rangle_{x \in W}$. Similarly, Propositions 3.6 and 2.6(c) imply that if γ and γ' are two game morphisms, then the action transformation of $\gamma' \circ \gamma$ is $\langle \alpha'_{\tau(x)} \circ \alpha(x) \rangle_{x \in W}$.

4. SPECIAL MORPHISMS

4.1. MONOMORPHISMS

Proposition 4.1. A CLT morphism $[\Theta, \Theta', \tau]$ is monic iff τ is injective. (Proof D.1.)

Proposition 4.1 characterizes a CLT monomorphism by the injectivity of τ . The injectivity of τ implies the injectivity of each action transformation α_x (Lemma D.2). Figure 4.1 provides an example of a CLT monomorphism (another example is Figure 4.4). HIDE²³

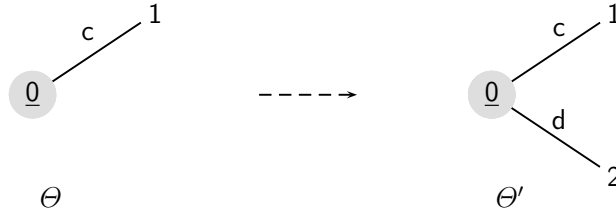


FIGURE 4.1. The CLT monomorphism $[\Theta, \Theta', \text{inc}_{X, X'}]$.²³

Proposition 4.2. A \mathbf{Gm} morphism $[\Gamma, \Gamma', \tau]$ is monic iff its run transformation ζ is injective. (Proof D.6.)

Proposition 4.2 characterizes a game monomorphism by the injectivity of its run transformation ζ . By ζ ’s definition (19), the injectivity of ζ is weaker than the injectivity of τ in Proposition 4.1, and further, the next paragraph will show by example that the injectivity of ζ is strictly weaker. The salient difference between Propositions 4.1 and 4.2 is that game morphisms are built on CLT morphisms that are end-preserving (Definition 3.4 condition (gZ)).

For an example, consider Figure 4.2. Here ζ is injective, since it maps the two source runs to the two target runs. Yet τ is not injective since $\tau(41) = \tau(42) = 40$. Thus

²³For sets C and D such that $C \subseteq D$, let $\text{inc}_{C, D}$ be the inclusion function from C to D .

Propositions 4.1 and 4.2 imply $[\Gamma, \Gamma', \tau]$ is monic but $[\Theta, \Theta', \tau]$ is not.²⁴ To develop intuition for $[\Gamma, \Gamma', \tau]$ being monic, let γ denote $[\Gamma, \Gamma', \tau]$ and consider the problem of specifying distinct $\gamma^1 = [\Gamma^*, \Gamma, \tau^1]$ and $\gamma^2 = [\Gamma^*, \Gamma, \tau^2]$ such that $\gamma \circ \gamma^1 = \gamma \circ \gamma^2$. The non-injectivity of τ suggests specifying an $x^* \in X^*$ such that $\tau^1(x^*) = 41$ and $\tau^2(x^*) = 42$. But since γ^1 and γ^2 must be built on end-preserving CLT morphisms, and since 41 and 42 are not end nodes, x^* cannot be an end node. This suggests specifying a successor y^* of x^* . But then (cE) for γ^1 implies $\tau^1(y^*) = 81$, and (cE) for γ^2 implies $\tau^2(y^*) = 82$. These equations and the definition of τ imply $\tau \circ \tau^1(y^*) = 81$ and $\tau \circ \tau^2(y^*) = 82$, which contradict $\gamma \circ \gamma^1 = \gamma \circ \gamma^2$.

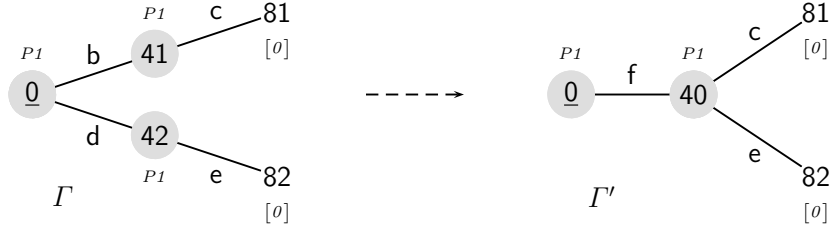


FIGURE 4.2. The game monomorphism $[\Gamma, \Gamma', \tau]$ defined by $\tau(x) = 40$ for $x \in \{41, 42\}$ and $\tau(x) = x$ otherwise. Its CLT morphism $[\Theta, \Theta', \tau]$ is not monic.

Game monomorphisms can also have non-injective action and player transformations. Figure 4.2's example has a non-injective action transformation because the definition of α (11) implies both $\alpha_0(\mathbf{b}) = \mathbf{f}$ and $\alpha_0(\mathbf{d}) = \mathbf{f}$. Meanwhile, Figure 4.3's example has a non-injective player transformation because definition of ι (21) implies both $\iota(P1) = \iota(\kappa(0)) = \kappa'(\tau(0)) = P1$ and $\iota(P2) = \iota(\kappa(1)) = \kappa'(\tau(1)) = P1$.

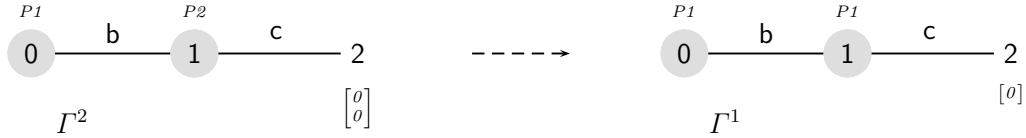


FIGURE 4.3. The game monomorphism $[\Gamma^2, \Gamma^1, \text{id}_{X^2}]$ (Figures 3.3 and 4.3 reverse Γ^1 and Γ^2). Its player transformation ι takes both $P1$ and $P2$ to $P1$.

4.2. ISOMORPHISMS

Proposition 4.3 uses the term “homeomorphism” from topology. If $[\Theta, \Theta', \tau]$ is a morphism and τ is bijective, then $\tau|_{W, W'}$ is a homeomorphism iff $(\tau|_{W, W'})^{-1}$ is continuous. For example, in Figure 4.4, $\tau|_{W, W'} = \text{id}_W$ is not a homeomorphism because $(\tau|_{W, W'})^{-1} = \text{id}_{W'}$ is not continuous. In particular, $(\tau|_{W, W'})^{-1} = \text{id}_{W'}$ splits the information set $\{3, 4\}$ (this characterization of discontinuity is from Proposition 2.2 applied to $[\Theta', \Theta, \tau^{-1}]$). For a more symmetric perspective, suppose we know only

²⁴This is consistent with Proposition 3.6's forgetful functor because a functor need not take a monomorphism to a monomorphism.

that Θ and Θ' are CLTs and that $\tau: X \rightarrow X'$ is a bijection. Then $\tau|_{W, W'}$ is a homeomorphism iff $\mathcal{H} \ni H \mapsto \bar{\tau}(H) \in \mathcal{H}'$ is a bijection (Lemma E.2). Thus a homeomorphism does not split information sets in either direction.

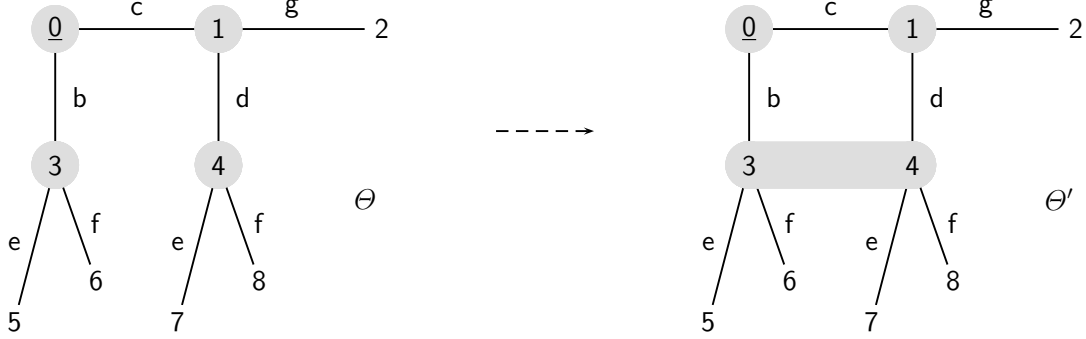


FIGURE 4.4. The morphism $[\Theta, \Theta', \text{id}_X]$ is not an isomorphism because $\tau|_{W, W'} = \text{id}_W$ is not a homeomorphism.

Proposition 4.3. *A CLT morphism $[\Theta, \Theta', \tau]$ is an isomorphism iff HIDE^{25}*

- (a) τ is a bijection²⁵ and
- (b) $\tau|_{W, W'}$ is a homeomorphism.

Further, if $[\Theta, \Theta', \tau]$ is an isomorphism, the following hold.

- (c) $\tau|_{X \setminus W, X' \setminus W'}$ is a bijection.
- (d) $E \ni xy \mapsto \tau(x)\tau(y) \in E'$ is a bijection.
- (e) $(\forall x \in X, y \in X) x \preceq y \Leftrightarrow \tau(x) \preceq' \tau(y)$.
- (f) $(\forall x \in X, y \in X) x \prec y \Leftrightarrow \tau(x) \prec' \tau(y)$.
- (g) $\tau(r) = r'$.
- (h) $(\forall y \in X) \bar{\tau}(P(y)) = P'(\tau(y))$.
- (i) $\bar{\tau}|_{\mathcal{H}, \mathcal{H}'}$ is a bijection.
- (j) In the action transformation $\langle \alpha_x \rangle_{x \in W}$ of $[\Theta, \Theta', \tau]$, each α_x is a bijection.
- (k) $[\Theta, \Theta', \tau]$ is end-preserving.
- (l) The run transformation ζ of $[\Theta, \Theta', \tau]$ is the bijection $\bar{\tau}|_{Z, Z'}$.
- (m) The inverse of $[\Theta, \Theta', \tau]$ is $[\Theta', \Theta, \tau^{-1}]$.
- (n) The action transformation of $[\Theta', \Theta, \tau^{-1}]$ is $\langle \alpha_{\tau^{-1}(x')}^{-1} \rangle_{x' \in W'}$.
- (o) $[\Theta', \Theta, \tau^{-1}]$ is end-preserving.
- (p) The run transformation of $[\Theta', \Theta, \tau^{-1}]$ is ζ^{-1} . (Proof E.5.)

To explore Proposition 4.3(a–b)’s characterization, this paragraph relaxes (a) and (b) independently. In Figure 4.1, τ is not bijective in violation of (a), even though $\tau|_{W, W'} = \text{id}_{\{0\}}$ is a homeomorphism in accord with (b). Here the tuple $[\Theta', \Theta, \tau^{-1}]$ cannot be constructed because τ^{-1} does not exist. In Figure 4.4, τ is bijective but $\tau|_{W, W'}$ is not a homeomorphism. Here the tuple $[\Theta', \Theta, \tau^{-1}] = [\Theta', \Theta, \text{id}_{X'}]$ is not a morphism because $\text{id}_{X'}$ is not continuous in violation of (cH) in Definition 2.4.

²⁵Condition (a) follows from condition (b) when neither Θ nor Θ' has an end node, that is, when $X \setminus W = \emptyset$ and $X' \setminus W' = \emptyset$. This happens when both Θ and Θ' have only infinite runs.

Proposition 4.4(c) below uses the term “preorder isomorphism”. This is the concept of order isomorphism from order theory, extended from orders to preorders (Davey and Priestley 2002, page 3; Viglizzo 2023). In particular, for each player $i \in I$, ζ is an preorder isomorphism from $(\mathcal{Z}, \lesssim_i)$ to $(\mathcal{Z}', \lesssim'_{i(i)})$ iff $\zeta: \mathcal{Z} \rightarrow \mathcal{Z}'$ is a bijection and both ζ and ζ^{-1} are monotone.

Proposition 4.4. *A \mathbf{Gm} morphism $[\Gamma, \Gamma', \tau]$ is an isomorphism iff HIDE^{26} HIDE^{27}*

- (a) $[\Theta, \Theta', \tau]$ is a **CLT** isomorphism,
- (b) ι is a bijection,²⁶ and
- (c) $(\forall i \in I) \zeta$ is a preorder isomorphism from $(\mathcal{Z}, \lesssim_i)$ to $(\mathcal{Z}', \lesssim'_{i(i)})$.²⁷

Further, if $[\Gamma, \Gamma', \tau]$ is an isomorphism, the following hold.

- (d) The inverse of $[\Gamma, \Gamma', \tau]$ is $[\Gamma', \Gamma, \tau^{-1}]$.
- (e) The player transformation of $[\Gamma', \Gamma, \tau^{-1}]$ is ι^{-1} . (Proof E.8.)

Since there is a forgetful functor from \mathbf{Gm} to \mathbf{CLT} (Proposition 3.6), any \mathbf{Gm} isomorphism has an underlying \mathbf{CLT} isomorphism in accord with Proposition 4.4(a). So, to explore Proposition 4.4(a–c)’s characterization, the remainder of this paragraph will relax conditions (b) and (c) independently. Figure 4.3 satisfies (a) and (c) but not (b). Here the tuple $[\Gamma', \Gamma, \tau^{-1}] = [\Gamma^1, \Gamma^2, \text{id}_{X^1}]$ is not a morphism because it violates (gK) of Definition 3.4—specifically, nodes 0 and 1 are controlled by the same player in Γ' but $\tau^{-1}(0)$ and $\tau^{-1}(1)$ are controlled by different players in Γ . Further, Figure 4.5 satisfies (a) and (b) but not (c). Here the tuple $[\Gamma', \Gamma, \text{id}_X]$ is not a morphism because it violates (gU) of Definition 3.4—specifically, $\{0, 2\} \lesssim'_{P_1} \{0, 1\}$ but not $\zeta^{-1}(\{0, 2\}) \lesssim_{P_1} \zeta^{-1}(\{0, 1\})$.

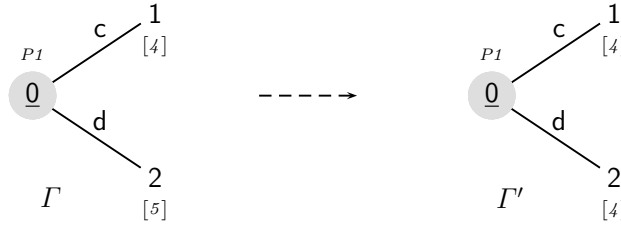


FIGURE 4.5. The morphism $[\Gamma, \Gamma', \text{id}_X]$ is not an isomorphism because of the utility functions. Specifically, $\{0, 1\} <_{P_1} \{0, 2\}$ and $\zeta(\{0, 1\}) \approx'_{P_1} \zeta(\{0, 2\})$.

Incidentally, all one-node CLTs are isomorphic to one another. To see this via Proposition 4.3(a–b)’s characterization, consider two one-node CLTs Θ and Θ' . Here the only conceivable transformation τ maps the one node of Θ to the one node of Θ' ,

²⁶Technically, condition (b) can be weakened to the injectivity of $\iota: I \rightarrow I'$ because condition (a) implies the surjectivity of ι . In particular, the image $\bar{\iota}(I)$ by ι 's definition (21) is equal to $\bar{\kappa}'(\bar{\tau}(W))$, which by Proposition 4.3(b) is equal to $\bar{\kappa}'(W')$, which by the definition of I' is equal to I' .

²⁷By the definitions (17) of $\langle \lesssim_i \rangle_{i \in I}$ and $\langle \lesssim'_{i'} \rangle_{i' \in I'}$, condition (c) is equivalent to $(\forall i \in I, Z_1 \in \mathcal{Z}, Z_2 \in \mathcal{Z}) U_i(Z_1) \leq U_i(Z_2) \Leftrightarrow U'_{\iota(i)} \circ \zeta(Z_1) \leq U'_{\iota(i)} \circ \zeta(Z_2)$. Also, condition (a) and Proposition 4.3(1) imply that ζ and $\bar{\tau}|_{\mathcal{Z}, \mathcal{Z}'}$ are interchangeable.

and the homeomorphism $\tau|_{W,W'}$ is the empty function. Further, all one-node games are isomorphic to one another. To see this via Proposition 4.4(a–c)’s characterization, recall that all one-node CLTs are isomorphic, and note that each one-node game has no players.

APPENDIX A. EXAMPLES

A.1. GENERAL ORIENTATION

Readers less unacquainted with game theory might begin with Figure A.1(a). The nodes are numbers, the edges are shown by line segments, the root node is 0, the information sets are shown by shaded regions, and the actions are letters. The players are $P1$, $P2$ and $P3$, and the utility (payoff) profiles are beneath the five terminal nodes.²⁸

It may be helpful to see a brief history of extensive-form game specifications. Broadly speaking, there are three kinds of nodes. (i) As in Figure A.1(f), von Neumann and Morgenstern 1944 (pages 73–74; 78) specified each node as a set of outcomes, and used set inclusion to arrange these nodes in a tree. (ii) As in Figure A.1(a,b,d), Kuhn 1953 specified each node as an abstract entity without any internal structure, and then separately specified the edges of the tree connecting the nodes. (iii) As in Figure A.1(c), Harris 1985 and Osborne and Rubinstein 1994 (page 200) specified each node as a sequence of past actions, and then used concatenation to arrange these sequences in a tree. That created the first tractable specification of infinite-horizon (that is, infinite-depth) extensive-form games. More recently, Alós-Ferrer and Ritzberger 2016 returned to the outcome-set style of von Neumann and Morgenstern 1944 and extended it to allow for an infinite horizon. Finally, Streufert 2019 showed how the sequences of Harris 1985 and Osborne and Rubinstein 1994 can be changed into sets, as in Figure A.1(e).

A.2. STYLE GROUPS

The examples of Figure A.1 represent five groups of specification styles. These five groups are discussed in the following five paragraphs. Most of these groups contain many different styles from different academic fields. Within this diverse literature, nodes are sometimes called “vertices”, “states”, “histories”, or “positions”, and actions are sometimes called “labels”, “alternatives”, “choices”, or “programs”.

²⁸This game is from Selten 1975, Figure 1, and is commonly known as the “horse” game simply because it looks like a stick-drawing of a horse. For a story for the game, suppose a student (called player $P1$) must decide between the bad action of not doing her homework (called b) and the correct action of doing her homework (called c). Knowing that the homework has been finished (node 1), a dog (player $P2$) must decide between the dumb action of eating the homework (d) and the good action of taking a nap (g). Finally, without knowing whether the student chose bad (node 3) or the student chose correct and the dog chose dumb (node 4), the teacher (player $P3$) must decide between excusing the student (e) and failing the student (f). The student most prefers being excused without doing the homework (run $\{0,3,5\}$), and least prefers failing after doing the homework (run $\{0,1,4,8\}$). The dog likes eating homework (runs $\{0,1,4,7\}$ and $\{0,1,4,8\}$). The teacher does not want to excuse a badly behaving student (run $\{0,3,5\}$) or to fail a correctly behaving student (run $\{0,1,4,8\}$).

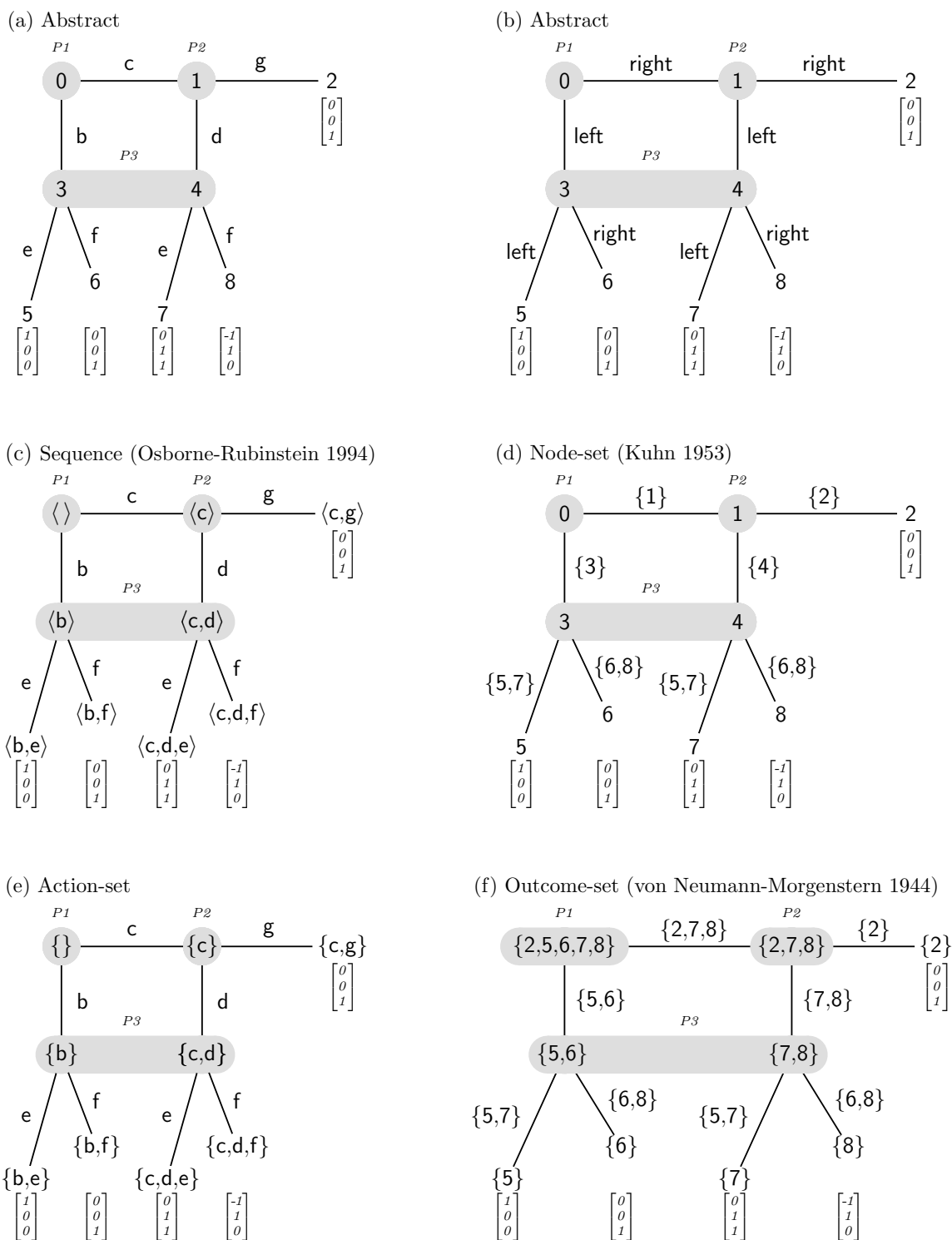


FIGURE A.1. Examples of different specification styles. Roughly, generic styles are in top row, backward-looking styles are lower left, and forward-looking styles are lower right. In these examples, utilities do not change.

Abstract (Figure A.1(a,b)). The styles in this group place no restrictions on the notation for nodes and actions. Examples from economics include the style in Kreps and Wilson 1982; the style in Myerson 1991, Section 2.1; and the style in Mas-Colell, Whinston, and Green 1995, page 227. Examples from elsewhere include the “labeled transition system” style in Blackburn, de Rijke, and Venema 2001, page 3; the style in Shoham and Leyton-Brown 2009, page 125; and the “epistemic process graph” style in van Benthem 2014, page 70.²⁹ A final example is the “**Gm**” style in this paper, which is virtually identical to the “traditional style” in Streufert 2025.³⁰ Since this group specifies nodes and actions abstractly, it can be understood to subsume the other four groups. Relatedly, this paper’s **Gm** games can be understood to include almost all extensive-form games.³¹

Sequence (Figure A.1(c)). The styles in this group specify actions abstractly and then specify each node as the sequence of past actions leading up to it. This group is arguably the most popular, and is ideally suited for repeated games and for stationary economic models in general. Specific examples from economics include the styles in Harris 1985 and Osborne and Rubinstein 1994, page 200. Examples from elsewhere include the “protocol” style in Parikh and Ramanujam 1985; the “history-based multi-agent structure” style in Pacuit 2007; the “sequence-form representation” style in Shoham and Leyton-Brown 2009, page 129; the style in Mann, Sandu, and

²⁹Some labeled transition systems and process graphs have recursive transitions. These do not support extensive-form games because extensive-form games require trees. Similarly, stochastic games, such as those in Mertens 2002, are not extensive-form games because they have recursive transitions.

³⁰The “games” of Definition 3.1 are more general than the “traditional games” of Streufert 2024 (Definition 5.1) to the extent that the definition here admits games in which X is a singleton (all of which have empty E , \mathcal{H} , λ , κ , and U). Otherwise, the differences between the definitions are cosmetic. Specifically, [1] this paper regards a function as a triple which specifies a codomain (note 7 here), while Streufert 2024 regards a function as a graph which does not specify a codomain (note 13 there). [2] This paper uses the language of topology to discuss conditions (9) and (16) here, while Streufert 2024 uses the language of measure theory to discuss the equivalent conditions (28) and (30) there. (Both [1] and [2] are meant to bring this paper closer to the category-theory literature.) [3] The labeling function (i.e., action-assigning function) is called λ here and α there. [4] The control-assigning function (i.e., player-assigning function) is called κ here and ι there. [5] In this paper, a continuously labeled tree (CLT) includes roughly half of the components of a game, and provides a useful intermediate point. In Streufert 2024, CLTs do not appear.

³¹**Gm** games include most but not all extensive-form games. [a] Although infinite runs are allowed, a **Gm** game is discrete in the sense that each node has a finite number of predecessors. This excludes non-discrete extensive-form games such as those in Dockner, Jørgensen, Long, and Sorger 2000, and Alós-Ferrer and Ritzberger 2016, Chapter 5. [b] Although a **Gm** game can specify a simultaneous move via multiple information sets (as in Osborne and Rubinstein 1994, page 202), it cannot specify multiple players moving at the same information set (as in Alós-Ferrer and Ritzberger 2016, page 140). [c] The many differences between members of the abstract group are being suppressed (details in notes 6, 10, 12, 19, and 20). [d] In a different direction, Streufert 2025 essentially shows there is a bijection between **Gm** games and pentaform games, which are formulated in a substantially different way. The author intends to build a category for pentaform games which is isomorphic to the category **Gm**.

Sevenster 2011, Chapter 2; the “logical game” style in Hodges 2013, Section 2; and the “epistemic forest model” style in van Benthem 2014, page 130.

Action-set (Figure A.1(e)). The styles in this group specify actions abstractly and then specify each node as the set (rather than the sequence) of past actions leading up to it. This small group includes the styles of Streufert 2015 and 2019. Its technical advantage is that a set of actions is a simpler mathematical object than a sequence of actions. These action-set nodes are “dual” to the outcome-set nodes of the outcome-set group in the sense that action sets look backward while outcome sets look forward.

Node-set (Figure A.1(d)). The styles in this group specify nodes abstractly and then specify each action as the set of nodes to which the action leads (or similarly, as the set of edges that end at the nodes to which the action leads). This group includes the style of Kuhn 1953, the style of Selten 1975, the “simple” style of Alós-Ferrer and Ritzberger 2016, Section 6.3, and the style of González-Díaz, García-Jurado, and Fiestras-Janeiro 2023, Section 3.1.

Outcome-set (Figure A.1(f)). The styles in this group specify both nodes and actions as sets of outcomes. This group includes the style in von Neumann and Morgenstern 1944, Section 10, as well as the style in Alós-Ferrer and Ritzberger, Section 6.2. Note that outcome-set nodes shrink as one goes forward, away from the root node. In this sense, the node collection of an outcome-set game resembles the filtration of a stochastic process.

A.3. DISTINGUISHED ACTIONS

The five-group catalog of Section A.2 can be complemented with a partial second “dimension”. In Figure A.1(b), the actions **left** and **right** are shared by the three information sets. Figure A.1(a) is different. There, the actions of the three information sets are distinguished from one another. Distinguished actions allow one to economize on notation and terminology. For example, a strategy in a general game is a function taking each decision node to the action to be played at that node. Alternatively, a strategy in a game with distinguished actions can be defined as merely the set of actions to be played. An early user of distinguished actions was Kreps and Wilson 1982 (page 867). Distinguished actions were also assumed throughout the “node-and-choice” categories of Streufert 2017, 2020a, and 2020b (distinguished actions are sometimes called “choices”).

Games in the abstract and sequence style groups of Section A.2 may or may not have distinguished actions. However, games in the action-set, node-set, and outcome-set groups must have distinguished actions because of specialized group properties (details in Streufert 2019, Section 7).

A.4. EQUIVALENCE BETWEEN STYLE GROUPS

One might conjecture that the five style groups are somehow equivalent, and indeed, this conjecture has been partially confirmed by the literature. Broadly speaking, Alós-Ferrer and Ritzberger 2016 showed that each game in a certain node-set style could be mapped to a game in a certain outcome-set style, and vice versa. Similarly, Kline and Luckraz 2016 showed that each game in a certain abstract style could be mapped

to a game in a certain sequence style, and vice versa. In a similar fashion, Streufert 2019 connected an action-set style with a sequence style, and an abstract style with a node-set style. Thus representative styles from all five style groups have been linked together.

However, the situation is nuanced. [1] The results in the previous paragraph concern only a representative style from each style group, and do not formally define the groups themselves. [2] Distinguished actions are required by some groups but not others (as noted in Section A.3). [3] Only the abstract and sequence groups are capable of expressing absentmindedness (Piccione and Rubinstein 1997), which is also known as nonlinearity (Isbell 1957). Thus the action-set, node-set, and outcome-set groups are less general in a substantial sense (details in Streufert 2019, Section 7). [4] The previous paragraph’s literature only establishes ad hoc functions between different styles, and so each function’s meaning rests solely on its own intuitive appeal. It is hoped that these ad hoc functions can be unified, justified, and extended by the isomorphisms of **Gm**. Indeed, Streufert 2021 (Section 5) has begun to do so, as discussed at the end of Section 1.2.

APPENDIX B. FOR DEFINITION OF CLT

Lemma B.1. *Suppose (X, E) is an out-tree (2), and $\lambda: E \rightarrow A$ is surjective and locally injective (6). Derive the feasibility correspondence F (7) and the next-node function n (8). Then the following hold.*

- (a) $\bigcup_{x \in W} F(x) = A$.
- (b) The function n is well-defined.
- (c) Fix a decision node $x_o \in W$. Then

$$F(x_o) \ni a \mapsto n(x_o, a) \in \{y_+ | x_o y_+ \in E\}$$

is a bijection. Its inverse is $F(x_o) \ni \lambda(x_o y) \leftarrow y \in \{y_+ | x_o y_+ \in E\}$.³²

Proof. (a). For the forward inclusion $\bigcup_{x \in W} F(x) \subseteq A$, note that each $F(x) \subseteq A$ by the definition (7) of F . For the reverse inclusion, take $a \in A$. Then the definition (5) of A implies there is [1] $xy \in E$ such that [2] $\lambda(xy) = a$. Then [1] and the definition (3) of W imply $x \in W$. Also [2] and the definition (7) of F imply $a \in F(x)$.

(b). By the definition (8) of n , it suffices to show that for each $(x, a) \in F^{\text{gr}}$, there is a unique $y \in X \setminus \{r\}$ such that $\lambda(xy) = a$. Toward that end, take $(x, a) \in F^{\text{gr}}$. Then by the definition of F , there is a y such that $\lambda(xy) = a$. Further, if y_* satisfies $\lambda(xy_*) = a$, then the local injectivity (6) of λ implies $y = y_*$. Finally, $y \in X \setminus \{r\}$ by the definition (2) of an out-tree.

(c). The forward function is well-defined by part (b), and the reverse function is well-defined since the domain of λ is E . Thus it suffices to show that [1] the reverse after the forward is $\text{id}_{F(x_o)}$, and that [2] the forward after the reverse is $\text{id}_{\{y_+ | x_o y_+ \in E\}}$.

For [1], take a feasible action $a \in F(x_o)$. It must be shown that $\lambda(x_o n(x_o, a)) = a$. This holds since $n(x_o, a)$ is defined by (8) to be the unique y_* that satisfies $\lambda(x_o y_*) = a$.

³²Intuitively, n and λ are “local inverses” at x_o .

For [2], take a node y that immediately follows x_o in the sense that $x_o y \in E$. The definition of $F(x_o)$ implies $\lambda(x_o y) \in F(x_o)$, which implies $(x_o, \lambda(x_o y)) \in F^{\text{gr}}$, which implies $(x_o, \lambda(x_o y))$ is in the domain of n . Thus the definition (8) of n implies $n(x_o, \lambda(x_o y))$ is the unique y_* such that $\lambda(x_o y_*) = \lambda(x_o y)$ [to see this, regard $\lambda(x_o y)$ as a constant]. This equality and local injectivity (6) imply $y_* = y$, which implies $n(x_o, \lambda(x_o y)) = y$. \square

Proof B.2 (for Proposition 2.2). By general topology, $\tau|_{W, W'}$ is continuous iff

$$(26) \quad (\forall x \in W, H' \in \mathcal{H}') \tau(x) \in H' \text{ implies } (\exists H \in \mathcal{H}) x \in H \text{ and } \bar{\tau}(H) \subseteq H'.$$

Thus it suffices to show that (26) is equivalent to

$$(27) \quad (\forall H \in \mathcal{H})(\exists H' \in \mathcal{H}') \bar{\tau}(H) \subseteq H'.$$

For the forward direction, assume (26) and take [A] $H \in \mathcal{H}$. Then (C2) for Θ implies there is [B] $x \in W$ such that [C] $x \in H$. Further $\bar{\tau}(W) \subseteq W'$ and (C2) for Θ' imply there is [D] $H' \in \mathcal{H}'$ such that [E] $\tau(x) \in H'$. Note [B], [D], [E] and (26) imply there is [F] $H_o \in \mathcal{H}$ such that [G] $x \in H_o$ and [H] $\bar{\tau}(H_o) \subseteq H'$. Further, [A] and [C], and [F] and [G], and (C2) for Θ imply $H = H_o$. Thus [H] implies $\bar{\tau}(H) \subseteq H'$.

For the reverse direction, assume (27), and take [A] $x \in W$ and [B] $H' \in \mathcal{H}'$ such that [C] $\tau(x) \in H'$. Note [A] and (C2) for Θ imply there is [D] $H \in \mathcal{H}$ such that [E] $x \in H$. So it suffices to show $\bar{\tau}(H) \subseteq H'$. Note [D] and (27) imply there is [F] $H_* \in \mathcal{H}'$ such that [G] $\bar{\tau}(H) \subseteq H_*$. Easily, [E] implies $\tau(x) \in \bar{\tau}(H)$, which by [G] implies [H] $\tau(x) \in H_*$. Further, [B] and [C], and [F] and [H], and (C2) for Θ' imply $H' = H_*$. Thus [G] implies $\bar{\tau}(H) \subseteq H'$. \square

Proof B.3 (for Proposition 2.3). Consider any decision node $x \in W$ and any feasible action $a \in F(x)$. The definition (8) of n implies that $\lambda(x n(x, a)) = a$, which implies that the edge satisfies $x n(x, a) \in E$, which by edge-preservation (10) implies $\tau(x) \tau(n(x, a)) \in E'$, which by the definition of the labeling function λ' implies $\lambda'(\tau(x) \tau(n(x, a)))$ exists. Thus since

$$(28) \quad \lambda'(\tau(x) \tau(n(x, a))) = \alpha_x(a)$$

by $\alpha_x(a)$'s definition (11), $\alpha_x(x)$ is well-defined. So it remains to show that $\alpha_x(a) \in F'(\tau(x))$. Consider the definition (7) of F' at $\tau(x) \in W'$. It implies that

$$F'(\tau(x)) = \{ a' \in A' \mid (\exists y') \lambda'(\tau(x) y') = a' \}.$$

By (28), the set's equality holds at $y' = \tau(n(x, a))$ and $a' = \alpha_x(a)$. So $\alpha_x(a)$ belongs to the set. \square

Proof B.4 (for Proposition 2.5). (a)–(e). These follow immediately from (cE). [Because of (cE), the function τ defines a graph morphism from (X, E) to (X', E') .]

(f). Take $xy \in E$. In steps, the definition (7) of F implies $\lambda(xy) \in F(x)$, which by the definition (11) of α_x implies

$$\alpha_x(\lambda(xy)) = \lambda'(\tau(x) \tau(n(x, \lambda(xy)))),$$

which by applying Lemma B.1(c) at $x_o = x$ reduces to $\alpha_x(\lambda(xy)) = \lambda'(\tau(x) \tau(y))$.

(g). Take $x \in W$ and $a \in F(x)$. The definition (11) of α_x implies

$$(29) \quad \lambda'(\tau(x) \tau(n(x, a))) = \alpha_x(a).$$

In addition, the definition of α_x implies $\alpha_x(a) \in F'(\tau(x))$, which easily implies $(\tau(x), \alpha_x(a)) \in (F')^{\text{gr}}$, which implies that $(\tau(x), \alpha_x(a))$ is in the domain of n . Thus the definition (8) of n' implies that $n'(\tau(x), \alpha_x(a))$ is the unique y' such that

$$\lambda'(\tau(x) y') = \alpha_x(a).$$

By (29), that y' is $\tau(n(x, a))$, and thus $n'(\tau(x), \alpha_x(a)) = \tau(n(x, a))$. \square

Lemma B.5. *Suppose Θ is a CLT. Then the action transformation (11) of the tuple $[\Theta, \Theta, \text{id}_X]$ is $\langle \text{id}_{F(x)} \rangle_{x \in W}$.*

Proof. Use (11) to construct the action transformation $\langle \alpha_x \rangle_{x \in W}$ of $[\Theta, \Theta', \tau] = [\Theta, \Theta, \text{id}_X]$. Take a decision node $x \in W$. Then definition (11) implies $\alpha_x: F(x) \rightarrow F'(\tau(x))$ which reduces to $\alpha_x: F(x) \rightarrow F(x)$. Thus it remains to show $(\forall a \in F(x)) \alpha_x(a) = a$. Toward that end, take a feasible action $a \in F(x)$. In steps, $\alpha_x(a)$ by definition (11) is $\lambda'(\tau(x) \tau(n(x, a)))$, which reduces to $\lambda'(x n(x, a))$, which reduces to $\lambda(x n(x, a))$, which by Lemma B.1(c) equals a . \square

Lemma B.6. *Suppose $[\Theta, \Theta', \tau]$ and $[\Theta', \Theta'', \tau']$ are morphisms. Then the action transformation (11) of the tuple $[\Theta, \Theta'', \tau' \circ \tau]$ is $\langle \alpha'_{\tau(x)} \circ \alpha_x \rangle_{x \in W}$.*

Proof. First consider the expression $\langle \alpha'_{\tau(x)} \circ \alpha_x \rangle_{x \in W}$ in the lemma. To show that

$$\langle \alpha'_{\tau(x)} \circ \alpha_x: F(x) \rightarrow F''(\tau' \circ \tau(x)) \rangle_{x \in W}$$

is well-defined, consider an individual decision node $x \in W$. Then $\alpha_x: F(x) \rightarrow F'(\tau(x))$ by definition (11) at x , and $\alpha'_{\tau(x)}: F'(\tau(x)) \rightarrow F''(\tau' \circ \tau(x))$ by the same definition (11) at $x' = \tau(x)$.

Second, let

$$\langle \alpha_x^*: F(x) \rightarrow F''(\tau' \circ \tau(x)) \rangle_{x \in W}$$

be the action transformation (11) derived from the tuple $[\Theta, \Theta'', \tau' \circ \tau]$. Note Definition 2.4 condition (cE) for the morphism $[\Theta, \Theta', \tau]$ implies $(\forall xy \in E) \tau(x) \tau(y) \in E'$. Thus (cE) for the morphism $[\Theta', \Theta'', \tau']$ implies $(\forall xy \in E) \tau' \circ \tau(x) \tau' \circ \tau(y) \in E''$. Hence Proposition 2.3 implies that $\langle \alpha_x^* \rangle_{x \in W}$ is well-defined.

Thus it remains to show that $(\forall a \in F(x)) \alpha_x^*(a) = \alpha'_{\tau(x)}(\alpha_x(a))$. Toward that end, take a feasible action $a \in F(x)$. Then $\alpha_x^*(a)$ by the definition (11) of α_x^* equals

$$\lambda''(\tau' \circ \tau(x) \tau' \circ \tau(n(x, a))),$$

which by Proposition 2.5(g) applied to $[\Theta, \Theta', \tau]$ equals

$$\lambda''(\tau'(\tau(x)) \tau'(n'(\tau(x), \alpha_x(a))))),$$

which by the definition (11) of $\alpha'_{\tau(x)}$ equals $\alpha'_{\tau(x)}(\alpha_x(a))$. \square

Proof B.7 (for Proposition 2.6). Since the identity laws and associativity hold by inspection, Claims 1(a) and 2(a) imply that **CLT** is a well-defined category. This proves the proposition's part (a). Its part (b) follows from Claim 1(b), and its part (c) follows from Claim 2(b).

Claim 1: Suppose Θ is a CLT. Then (a) id_Θ is a morphism and (b) id_Θ 's action transformation is $\langle \text{id}_{F(x)} \rangle_{x \in W}$. Definition (14) sets id_Θ equal to the tuple $[\Theta, \Theta, \text{id}_X]$. Thus Lemma B.5 shows that id_Θ 's action transformation is $\langle \text{id}_{F(x)} \rangle_{x \in W}$. This proves (b). Hence it remains to show that $[\Theta, \Theta, \text{id}_X]$ satisfies (cE), (cH), and (cL) of Definition 2.4. The first two hold by inspection. For (cL), note (b) implies the action transformation is $\langle \text{id}_{F(x)} \rangle_{x \in W}$, and this is continuous by (C4) of Definition 2.1.

Claim 2: Suppose $\theta = [\Theta, \Theta', \tau]$ and $\theta' = [\Theta', \Theta'', \tau']$ are morphisms. Then (a) $\theta' \circ \theta$ is a morphism and (b) $\theta' \circ \theta$'s action transformation is $\langle \alpha'_{\tau(x)} \circ \alpha_x \rangle_{x \in W}$. Definition 14 sets $\theta' \circ \theta$ equal to the tuple $[\Theta, \Theta'', \tau' \circ \tau]$. Thus Lemma B.6 shows that $\theta' \circ \theta$'s action transformation is $\langle \alpha'_{\tau(x)} \circ \alpha_x \rangle_{x \in W}$. This proves (b). Hence it remains to show that the tuple $[\Theta, \Theta'', \tau' \circ \tau]$ satisfies (cE), (cH), and (cL) from Definition 2.4. In this context, (cE) for $[\Theta, \Theta'', \tau' \circ \tau]$ is

$$(\forall xy \in E) \tau' \circ \tau(x) \tau' \circ \tau(y) \in E''.$$

To see this, take $xy \in E$. Then (cE) for θ implies $\tau(x)\tau(y) \in E'$, which by (cE) for θ' implies $\tau' \circ \tau(x) \tau' \circ \tau(y) \in E''$. Further, (cH) for $[\Theta, \Theta'', \tau' \circ \tau]$ is

$$(\tau' \circ \tau)|_{W, W''} \text{ is continuous.}$$

To see this, note that $\tau|_{W, W'}$ is continuous by (cH) for θ , and that $\tau'|_{W', W''}$ is continuous by (cH) for θ' . Thus $(\tau' \circ \tau)|_{W, W''} = \tau'|_{W', W''} \circ \tau|_{W, W'}$ is continuous. Finally, this claim's part (b) implies that (cL) for $[\Theta, \Theta'', \tau' \circ \tau]$ is

$$\langle \alpha'_{\tau(x)} \circ \alpha_x \rangle_{x \in W} \text{ is continuous from } W.$$

To see this, suppose x_1 and x_2 are in the same member of \mathcal{H} . By (cL) for θ , this assumption implies [1] $\alpha_{x_1} = \alpha_{x_2}$. By Proposition 2.2, this assumption also implies $\tau(x_1)$ and $\tau(x_2)$ are in the same member of \mathcal{H}' , which by (cL) for θ' implies [2] $\alpha'_{\tau(x_1)} = \alpha'_{\tau(x_2)}$. Finally, [1] and [2] imply $\alpha'_{\tau(x_1)} \circ \alpha_{x_1} = \alpha'_{\tau(x_2)} \circ \alpha_{x_2}$. \square

APPENDIX C. FOR DEFINITION OF **Gm**

Proof C.1 (for Proposition 3.2). Take any run $Z \in \mathcal{Z}$. By the definition (19) of ζ , it suffices to show that $P' \circ \tau(r) \cup \bar{\tau}(Z) \in \mathcal{Z}'$. Before beginning, note that the definition of P' implies

$$(30) \quad P' \circ \tau(r) \cup \{\tau(r)\} \text{ is the walk from } r' \text{ to } \tau(r).$$

On the one hand, suppose $Z \in \mathcal{Z}_{\text{ft}}$. Then there is a [1] $y \in X \setminus W$ such that [2] Z is the walk from r to y . Note [1] and end-preservation (18) imply [3] $\tau(y) \in X' \setminus W'$. Meanwhile, [2] and (cE) from Definition 2.1 imply that $\bar{\tau}(Z)$ is the walk from $\tau(r)$ to $\tau(y)$, which by (30) implies that the concatenation $P' \circ \tau(r) \cup \bar{\tau}(Z)$ is the walk from r' to $\tau(y)$. Thus [3] implies $P' \circ \tau(r) \cup \bar{\tau}(Z) \in \mathcal{Z}'_{\text{ft}}$.

On the other hand, take $Z \in \mathcal{Z}_{\text{inf}}$. Then Z is an infinite walk from r . Thus (cE) implies $\bar{\tau}(Z)$ is an infinite walk from $\tau(r)$, which by (30) implies that the concatenation $P' \circ \tau(r) \cup \bar{\tau}(Z)$ is an infinite walk from r' . Hence $P' \circ \tau(r) \cup \bar{\tau}(Z) \in \mathcal{Z}'_{\text{inf}}$. \square

Proof C.2 (for Proposition 3.3). (a). The set ι^{gr} (21) is a well-defined subset of $I \times I'$ because the definitions of κ and I imply $\kappa: W \rightarrow I$, and because the definitions of κ' and I' imply $\kappa': W' \rightarrow I'$. The projection of ι^{gr} on its first coordinate is I because κ is surjective by (G2). Thus it remains to show that each $i \in I$ is paired by ι^{gr} with exactly one $i' \in I'$. Toward that end, suppose (i, i'_1) and (i, i'_2) were two members of ι^{gr} . Then the definition (21) of ι^{gr} implies there are $x_1 \in W$ and $x_2 \in W$ such that

$$(i, i'_1) = (\kappa(x_1), \kappa'(\tau(x_1))) \text{ and } (i, i'_2) = (\kappa(x_2), \kappa'(\tau(x_2))).$$

The first components of these two equalities imply that $\kappa(x_1) = \kappa(x_2)$, which by control preservation (20) implies $\kappa' \circ \tau(x_1) = \kappa' \circ \tau(x_2)$, which by the second components of the equalities implies that $i'_1 = i'_2$.

(b). By inspection, the definition (21) of ι implies that $(\forall x \in W) \iota(\kappa(x)) = \kappa'(\tau(x))$. The composition $\iota \circ \kappa$ is well-defined because I is both the codomain of κ by (15), and the domain of ι by (21). The composition $\kappa' \circ \tau|_{W, W'}$ is well-defined because W' is both the codomain of $\tau|_{W, W'}$, and the domain of κ' by (G2). Further, W is the domain of both $\iota \circ \kappa$ by (G2), and $\kappa' \circ \tau|_{W, W'}$. Finally, I' is the codomain of both $\iota \circ \kappa$ by (21), and $\kappa' \circ \tau|_{W, W'}$ by (15). \square

Lemma C.3. *Suppose Γ is a game and consider the tuple $[\Gamma, \Gamma, \text{id}_X]$. Then (a) its run transformation (19) is $\text{id}_{\mathcal{Z}}$, and (b) its player transformation (21) is id_I .*

Proof. (a). Use definition (19) to construct the run transformation ζ of $[\Gamma, \Gamma', \tau] = [\Gamma, \Gamma, \text{id}_X]$. Then definition (19) implies $\zeta: \mathcal{Z} \rightarrow \mathcal{Z}'$, which reduces to $\zeta: \mathcal{Z} \rightarrow \mathcal{Z}$. Thus it remains show $(\forall Z \in \mathcal{Z}) \zeta(Z) = Z$. Toward that end, take a run $Z \in \mathcal{Z}$. Then $\zeta(Z)$ by definition (19) is $P' \circ \tau(r) \cup \bar{\tau}(Z)$ which by $\tau = \text{id}_X$ reduces to $P'(r) \cup Z$, which reduces to $P(r) \cup Z$, which by inspection is Z .

(b). Use definition (21) to construct the player transformation ι of $[\Gamma, \Gamma', \tau] = [\Gamma, \Gamma, \text{id}_X]$. Then definition (21) implies $\iota: I \rightarrow I'$, which reduces to $\iota: I \rightarrow I$. Thus it remains to show $(\forall i \in I) \iota(i) = i$. Toward that end, take a player $i \in I$. By the definition (15) of I , there is $x \in W$ such that $\kappa(x) = i$. In steps, $\iota(i)$ by the local definition of x is $\iota(\kappa(x))$, which by the definition (21) of ι is $\kappa'(\tau(x))$, which reduces to $\kappa(\tau(x))$, which reduces to $\kappa(x)$, which by the local definition of x is i . \square

Lemma C.4. *Suppose $[\Theta, \Theta', \tau]$ and $[\Theta', \Theta'', \tau']$ are end-preserving (18) morphisms. Then (a) $[\Theta, \Theta'', \tau' \circ \tau]$ is an end-preserving morphism, and (b) its run transformation (19) is $\zeta' \circ \zeta$.*

Proof. Parts (a) and (b) follow from Claims 1 and 4, respectively.

Claim 1: $[\Theta, \Theta'', \tau' \circ \tau]$ is an end-preserving morphism. Proposition 2.6(a) (the well-definition of **CLT**) implies $[\Theta, \Theta'', \tau' \circ \tau]$ is a morphism. For end preservation (18), take $x \in X \setminus W$. Since $[\Theta, \Theta', \tau]$ is end-preserving, $\tau(x) \in X' \setminus W'$. Thus since $[\Theta', \Theta'', \tau']$ is end-preserving, $\tau' \circ \tau(x) \in X'' \setminus W''$.

Claim 2: $P'' \circ \tau'(r') \cup \bar{\tau}'(P' \circ \tau(r) \cup \{\tau(r)\}) = P'' \circ \tau' \circ \tau(r) \cup \{\tau' \circ \tau(r)\}$.

First, the definition of P'' implies

$$P'' \circ \tau'(r') \cup \{\tau'(r')\} \text{ is the walk in } (X'', E'') \text{ from } r'' \text{ to } \tau'(r'),$$

Second, the definition of P' implies $P' \circ \tau(r) \cup \{\tau(r)\}$ is the walk in (X', E') from r' to $\tau(r)$, which by (cE) for $[\Theta', \Theta'', \tau']$ implies

$$\bar{\tau}'(P' \circ \tau(r) \cup \{\tau(r)\}) \text{ is the walk in } (X'', E'') \text{ from } \tau'(r') \text{ to } \tau' \circ \tau(r).$$

The above imply that the concatenation $P'' \circ \tau'(r') \cup \bar{\tau}'(P' \circ \tau(r) \cup \{\tau(r)\})$ is the walk in (X'', E'') from r'' to $\tau' \circ \tau(r)$. Yet by the definition of P'' , we know $P'' \circ \tau' \circ \tau(r) \cup \{\tau' \circ \tau(r)\}$ is the walk from r'' to $\tau' \circ \tau(r)$. Thus the two walks are equal.

Claim 3: $(\forall Z \in \mathcal{Z}) P'' \circ \tau'(r') \cup \bar{\tau}'(P' \circ \tau(r) \cup \bar{\tau}(Z)) = P'' \circ \tau' \circ \tau(r) \cup \bar{\tau}' \circ \tau(Z)$.

Consider a run $Z \in \mathcal{Z}$. Because the root r is in Z , the left-hand side equals

$$P'' \circ \tau'(r') \cup \bar{\tau}'(P' \circ \tau(r) \cup \{\tau(r)\} \cup \bar{\tau}(Z)),$$

which by manipulation equals

$$P'' \circ \tau'(r') \cup \bar{\tau}'(P' \circ \tau(r) \cup \{\tau(r)\}) \cup \bar{\tau}' \circ \tau(Z),$$

which by Claim 2 equals

$$P'' \circ \tau' \circ \tau(r) \cup \{\tau' \circ \tau(r)\} \cup \bar{\tau}' \circ \tau(Z),$$

which by $r \in Z$ equals the right-hand side.

Claim 4: The run transformation of $[\Theta, \Theta'', \tau' \circ \tau]$ is $\zeta' \circ \zeta$. Since the morphisms $[\Theta, \Theta', \tau]$ and $[\Theta', \Theta'', \tau']$ are end-preserving by assumption, Proposition 3.2 implies $\zeta: \mathcal{Z} \rightarrow \mathcal{Z}'$ and $\zeta': \mathcal{Z}' \rightarrow \mathcal{Z}''$ are well-defined. Thus $\zeta' \circ \zeta: \mathcal{Z} \rightarrow \mathcal{Z}''$ is well-defined. Now let $\zeta^*: \mathcal{Z} \rightarrow \mathcal{Z}''$ denote the run transformation of $[\Theta, \Theta'', \tau' \circ \tau]$. Claim 1 and Proposition 3.2 imply $\zeta^*: \mathcal{Z} \rightarrow \mathcal{Z}''$ is well-defined. So it remains to show that $(\forall Z \in \mathcal{Z}) \zeta' \circ \zeta(Z) = \zeta^*(Z)$. This follows from Claim 3, the definition (19) of ζ , the definition (19) of ζ' , and the definition (19) of ζ^* . \square

Lemma C.5. *Suppose that Γ, Γ' , and Γ'' are games, that $[\Theta, \Theta', \tau]$ and $[\Theta', \Theta'', \tau']$ are CLT morphisms, and that the tuples $[\Gamma, \Gamma', \tau]$ and $[\Gamma', \Gamma'', \tau']$ are control-preserving (20). Then (a) $[\Gamma, \Gamma'', \tau' \circ \tau]$ is control-preserving (20) and (b) its player transformation (21) is $\iota' \circ \iota$.*

Proof. (a). By definition (20), it suffices to show

$$(\forall x_1 \in W, x_2 \in W) \kappa(x_1) = \kappa(x_2) \text{ implies } \kappa''(\tau' \circ \tau(x_1)) = \kappa''(\tau' \circ \tau(x_2)).$$

Toward that end, take decision nodes $x_1 \in W$ and $x_2 \in W$ such that $\kappa(x_1) = \kappa(x_2)$. Then $[\Gamma, \Gamma', \tau]$ preserving control (20) implies $\kappa'(\tau(x_1)) = \kappa'(\tau(x_2))$. Further, Proposition 2.5(e) implies $\bar{\tau}(W) \subseteq W'$, so the previous sentence and $[\Gamma', \Gamma'', \tau']$ preserving control (20) imply $\kappa''(\tau' \circ \tau(x_1)) = \kappa''(\tau' \circ \tau(x_2))$.

(b). First note $\iota' \circ \iota: I \rightarrow I''$, because $\iota: I \rightarrow I'$ and $\iota': I' \rightarrow I''$. Thus by definition (21), it suffices to show $(\iota' \circ \iota) \circ \kappa = \kappa'' \circ (\tau' \circ \tau)|_{W, W''}$. In steps, $\iota' \circ \iota \circ \kappa$ by Proposition 3.3(b) for $[\Gamma, \Gamma', \tau]$ equals $\iota' \circ \kappa' \circ \tau|_{W, W'}$, which by Proposition 3.3(b) for $[\Gamma', \Gamma'', \tau']$ equals $\kappa'' \circ \tau'|_{W', W''} \circ \tau|_{W, W'}$, which by inspection equals $\kappa'' \circ (\tau' \circ \tau)|_{W, W''}$. \square

Proof C.6 (for Proposition 3.5). Since the identity laws and associativity hold by inspection, Claims 1(a) and 2(a) imply that \mathbf{Gm} is a well-defined category. This proves the proposition's part (a). Its part (b) follows from Claim 1(b,c), and its part (c) follows from Claim 2(b,c).

Claim 1: Suppose Γ is a game. Then (a) id_Γ is a morphism, (b) its run transformation is id_Z , and (c) its player transformation is id_I . Definition (23) sets id_Γ equal to the tuple $[\Gamma, \Gamma, \text{id}_X]$. Thus Lemma C.3 shows that its run transformation is id_Z and its player transformation is id_I . This proves (b) and (c). Hence it remains to show that $[\Gamma, \Gamma, \text{id}_X]$ satisfies (gZ), (gK), and (gU) of Definition 3.4. In this context, (gZ) is

$$[\Theta, \Theta, \text{id}_x] \text{ is an end-preserving (18) CLT morphism.}$$

This holds because definition (14) implies $[\Theta, \Theta, \text{id}_X]$ is the CLT morphism id_X , and because end preservation (18) holds vacuously. Further, (gK) reduces to

$$[\Gamma, \Gamma, \text{id}_X] \text{ is control-preserving (20),}$$

which holds vacuously. Finally, this claim's parts (b) and (c) reduce (gU) to

$$(\forall i \in I) \text{id}_Z \text{ is monotone from } (\mathcal{Z}, \lesssim_i) \text{ to } (\mathcal{Z}, \lesssim_i),$$

which holds vacuously.

Claim 2: Suppose $\gamma = [\Gamma, \Gamma', \tau]$ and $\gamma' = [\Gamma', \Gamma'', \tau']$ are morphisms. Then (a) $\gamma' \circ \gamma$ is a morphism, (b) its run transformation is $\zeta' \circ \zeta$, and (c) its player transformation is $\iota' \circ \iota$. First consider $\gamma = [\Gamma, \Gamma', \tau]$ and $\gamma' = [\Gamma', \Gamma'', \tau']$ without composing them. Since both are morphisms, Definition 3.4 implies that both $[\Theta, \Theta', \tau]$ and $[\Theta', \Theta'', \tau']$ are end-preserving morphisms, and that both γ and γ' are control-preserving. Thus all the assumptions of Lemmas C.4 and C.5 are met.

Definition (23) sets $\gamma' \circ \gamma$ equal to $[\Gamma, \Gamma'', \tau' \circ \tau]$. Thus Lemma C.4(b) shows that its run transformation is $\zeta' \circ \zeta$, and Lemma C.5(b) shows that its player transformation is $\iota' \circ \iota$. This proves (b) and (c). Hence it remains to show that $[\Gamma, \Gamma'', \tau' \circ \tau]$ satisfies conditions (gZ), (gK), and (gU) of Definition 3.4. Lemma C.4(a) implies $[\Gamma, \Gamma'', \tau' \circ \tau]$ satisfies (gZ). Lemma C.5(a) implies $[\Gamma, \Gamma'', \tau' \circ \tau]$ satisfies (gK). Finally, (gU) for γ and (gU) for γ' imply

$$(\forall i \in I) \zeta' \circ \zeta \text{ is monotone from } (\mathcal{Z}, \lesssim_i) \text{ to } (\mathcal{Z}'', \lesssim''_{\iota' \circ \iota(i)}).$$

Hence this claim's parts (b) and (c) imply that $[\Gamma, \Gamma'', \tau' \circ \tau]$ satisfies (gU). \square

Proof C.7 (for Proposition 3.6). By condition (G1) of the game Definition 3.1, F_0 maps any game to a CLT. By condition (gZ) of the game-morphism Definition 3.4, F_1 maps any game morphism to a CLT morphism. By inspection, F preserves source and target. Thus the proposition follows from Claims 1 and 2.

Claim 1: F preserves identity. By the definition of id in \mathbf{Gm} (23), the image $F_1(\text{id}_\Gamma)$ is equal to

$$F_1([\Gamma, \Gamma, \text{id}_X]),$$

which by the definition of F_1 in the proposition is equal to

$$[F_0(\Gamma), F_0(\Gamma), \text{id}_X],$$

which by the definition of id in **CLT** (14), is equal to the identity $\text{id}_{F_0(\Gamma)}$.

Claim 2: F preserves composition. By the definition of \circ in **Gm** (23), the image $F_1([[\Gamma', \Gamma'', \tau'] \circ [\Gamma, \Gamma', \tau]])$ is equal to

$$F_1([\Gamma, \Gamma'', \tau' \circ \tau]),$$

which by the definition of F_1 in the proposition is equal to

$$[F_0(\Gamma), F_0(\Gamma''), \tau' \circ \tau],$$

which by the definition of \circ in **CLT** (14) is equal to

$$[F_0(\Gamma'), F_0(\Gamma''), \tau'] \circ [F_0(\Gamma), F_0(\Gamma'), \tau],$$

which by two applications of the proposition's definition of F_1 is equal to the composition $F_1([\Gamma', \Gamma'', \tau']) \circ F_1([\Gamma, \Gamma', \tau])$. \square

APPENDIX D. FOR MONOMORPHISMS

Proof D.1 (for Proposition 4.1). Consider a morphism $[\Theta, \Theta', \tau]$. By inspection, $[\Theta, \Theta', \tau]$ is monic if τ is injective. To show the converse, suppose that τ is not injective. Then there are $x_1 \in X$ and $x_2 \in X$ such that [A] $x_1 \neq x_2$ and [B] $\tau(x_1) = \tau(x_2)$.

In accord with the last sentences of Section 2.3, let $\Theta^* = [X^*, E^*, \mathcal{H}^*, \lambda^*]$ be a one-node CLT in which $X^* = \{r^*\}$, $E^* = \emptyset$, $\mathcal{H}^* = \emptyset$, and λ^* is the empty function. Then construct the tuple $[\Theta^*, \Theta, \tau^1]$ in which $\tau^1(r^*) = x_1$. This tuple is a morphism because (cE) holds by $E^* = \emptyset$, and (cH) and (cL) hold by $W^* = \emptyset$. Similarly construct the morphism $[\Theta^*, \Theta, \tau^2]$ in which $\tau^2(r^*) = x_2$.

The morphisms $[\Theta^*, \Theta, \tau^1]$ and $[\Theta^*, \Theta, \tau^2]$ are distinct because $\tau^1(r^*) = x_1$ and $\tau^2(r^*) = x_2$ are distinct by [A]. Yet the composition $[\Theta, \Theta', \tau] \circ [\Theta^*, \Theta, \tau^1]$ equals the composition $[\Theta, \Theta', \tau] \circ [\Theta^*, \Theta, \tau^2]$ because $\tau \circ \tau^1(r^*) = \tau(x_1)$ and $\tau \circ \tau^2(r^*) = \tau(x_2)$ are equal by [B]. Thus $[\Theta, \Theta', \tau]$ is not monic. \square

Lemma D.2. *Suppose $[\Theta, \Theta', \tau]$ is a monomorphism, and let $\langle \alpha_x \rangle_{x \in W}$ be its action transformation. Then $(\forall x \in W) \alpha_x$ is injective.*

Proof. By Proposition 4.1, τ is injective. Now take a decision node $x \in W$ and feasible actions $a_1 \in F(x)$ and $a_2 \in F(x)$. It suffices to show that $\alpha_x(a_1) = \alpha_x(a_2)$ implies $a_1 = a_2$. Toward that end, suppose $\alpha_x(a_1) = \alpha_x(a_2)$. Then α_x 's definition (11) implies

$$\lambda'(\tau(x) \tau(n(x, a_1))) = \lambda'(\tau(x) \tau(n(x, a_2))).$$

Thus λ' 's local injectivity (C3) implies $\tau(n(x, a_1)) = \tau(n(x, a_2))$. Thus the injectivity of τ implies $n(x, a_1) = n(x, a_2)$. Thus Lemma B.1(c) at $x_o = x$ implies $a_1 = a_2$. \square

Lemma D.3. *Suppose $\gamma = [\Gamma, \Gamma', \tau]$ is a morphism. Then γ is monic if its run transformation ζ is injective.*

Proof. To show the contrapositive, suppose γ is not monic. Then there are γ^1 and γ^2 such that $\gamma \circ \gamma^1 = \gamma \circ \gamma^2$ and $\gamma^1 \neq \gamma^2$. The equality implies that γ^1 and γ^2 have a common source, so denote that source Γ^* and let $\gamma^1 = [\Gamma^*, \Gamma, \tau^1]$ and $\gamma^2 = [\Gamma^*, \Gamma, \tau^2]$. Then the definition of γ^1 and γ^2 implies

$$(31) \quad \tau \circ \tau^1 = \tau \circ \tau^2.$$

and $\tau^1 \neq \tau^2$. This inequality implies there is $x^* \in X^*$ such that

$$(32) \quad \tau^1(x^*) \neq \tau^2(x^*).$$

Further, since any node is in at least one run, there is $Z^* \in \mathcal{Z}^*$ such that

$$(33) \quad x^* \in Z^*.$$

Note (31) and two applications of Proposition 3.5(b) imply $\zeta(\zeta^1(Z^*)) = \zeta(\zeta^2(Z^*))$. Thus to show that ζ is not injective, it suffices to show that $\zeta^1(Z^*) \neq \zeta^2(Z^*)$. This will be done by Claim 2.

Claim 1: Neither $\tau^1(x^*) \prec \tau^2(x^*)$ nor $\tau^1(x^*) \succ \tau^2(x^*)$. By symmetry, it suffices to show the first is impossible. If $\tau^1(x^*) \prec \tau^2(x^*)$, then Proposition 2.5(b) for γ would imply $\tau \circ \tau^1(x^*) \prec \tau \circ \tau^2(x^*)$, which would contradict (31).

Claim 2: $\zeta^1(Z^*) \neq \zeta^2(Z^*)$. To see this, suppose $\zeta^1(Z^*) = \zeta^2(Z^*)$. Then the definition (19) of ζ implies $P \circ \tau^1(r^*) \cup \overline{\tau^1}(Z^*) = P \circ \tau^2(r^*) \cup \overline{\tau^2}(Z^*)$, which by (33) implies

$$\tau^1(x^*) \in P \circ \tau^2(r^*) \cup \overline{\tau^2}(Z^*).$$

On the one hand, suppose $\tau^1(x^*) \in P \circ \tau^2(r^*)$. Then $\tau^1(x^*) \prec \tau^2(r^*)$. Meanwhile, $r^* \preceq^* x^*$ and Proposition 2.5(a) for γ^2 imply $\tau^2(r^*) \preceq \tau^2(x^*)$. The last two sentences imply $\tau^1(x^*) \prec \tau^2(x^*)$, which contradicts Claim 1.

On the other hand, suppose $\tau^1(x^*) \in \overline{\tau^2}(Z^*)$. Then (33) implies

$$\text{both } \tau^1(x^*) \text{ and } \tau^2(x^*) \text{ belong to } \overline{\tau^2}(Z^*).$$

Since Z^* is a walk (1) in (X^*, E^*) , Definition 2.4 condition (cE) for γ^2 implies $\overline{\tau^2}(Z^*)$ is a walk in (X, E) . Further, since a walk in an out-tree visits each of this nodes exactly once, $(\overline{\tau^2}(Z^*), \preceq)$ is a chain. Thus $\tau^1(x^*)$ and $\tau^2(x^*)$ belong to the same chain, which contradicts the combination of (32) and Claim 1. \square

Lemma D.4 (from graph theory). *Suppose that (X, E) and (X', E') are out-trees, that $\tau: X \rightarrow X'$ satisfies $(\forall xy \in E) \tau(x)\tau(y) \in E'$, and that N is a walk in (X, E) (expressed as a node set). Then (a) $\bar{\tau}(N)$ is a walk in (X', E') and (b) $\tau|_{N, \bar{\tau}(N)}$ is a graph isomorphism³³ from the out-tree $(N, E \cap N^2)$ to the out-tree $(\bar{\tau}(N), E' \cap \bar{\tau}(N)^2)$.*

Lemma D.5. *Suppose $\gamma = [\Gamma, \Gamma', \tau]$ is a morphism. Then if γ is monic, its run transformation ζ is injective.*

³³A graph isomorphism from one out-tree (X^A, E^A) to another out-tree (X^B, E^B) is a bijection $\tau: X^A \rightarrow X^B$ such that $\{\tau(x)\tau(y) \mid xy \in E^A\} = E^B$. (Streufert 2018 Section 2.5 develops the natural category for out-trees, and shows it is isomorphic to a full subcategory of **Grph** (Mac Lane 1998, page 48).)

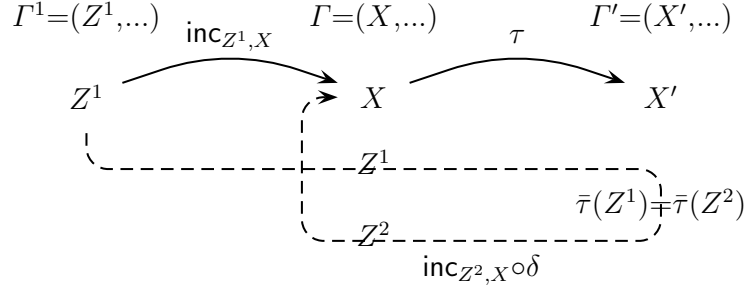


FIGURE D.1. The three node transformations of Lemma D.5's proof. Both Z^1 and Z^2 are subsets of X , and $\delta:Z^1 \rightarrow Z^2$ is $(\tau|_{Z^2, \bar{\tau}(Z^2)})^{-1} \circ \tau|_{Z^1, \bar{\tau}(Z^1)}$.

Proof. To prove the contrapositive, suppose ζ is not injective. Then there are runs $Z^1 \in \mathcal{Z}$ and $Z^2 \in \mathcal{Z}$ such that

$$(34a) \quad Z^1 \neq Z^2 \text{ and}$$

$$(34b) \quad \zeta(Z^1) = \zeta(Z^2).$$

By \mathcal{Z} 's definition (4), Z^1 and Z^2 are walks (1) in (X, E) which are expressed as node sets. Let

$$(35) \quad E^1 = E \cap (Z^1)^2 \text{ and } E^2 = E \cap (Z^2)^2$$

be their respective edge sets, so that (Z^1, E^1) and (Z^2, E^2) are the same walks expressed as out-trees.

Define the tuple

$$(36) \quad \Gamma^1 = (Z^1, E^1, \mathcal{H}^1, \lambda^1, \kappa^1, U^1)$$

by augmenting (Z^1, E^1) as follows. First define the decision-node set $W^1 = \pi_1 E^1$ and the partition $\mathcal{H}^1 = \{ \{x\} \mid x \in W^1 \}$. Second define the labeling function $\lambda^1: E^1 \rightarrow \{\mathbf{go}\}$ by $(\forall xy \in E^1) \lambda^1(xy) = \mathbf{go}$ (the only action is to “go” along the walk Z^1). Third define the control-assigning function $\kappa^1 = \text{id}_{W^1}$ (this unusual construction identifies the player set I^1 with the decision-node set W^1). Finally define $U^1 = \langle U_i^1: \{Z^1\} \rightarrow \bar{\mathbb{R}} \rangle_{i \in W^1}$ by $(\forall i \in W^1) U_i^1(Z^1) = 0$ (each U_i^1 has a singleton domain).

Next, define the tuple

$$(37) \quad \gamma^1 = [\Gamma^1, \Gamma, \text{inc}_{Z^1, X}],$$

which is well-defined since $Z^1 \subseteq X$. Further, define

$$(38) \quad \delta = (\tau|_{Z^2, \bar{\tau}(Z^2)})^{-1} \circ \tau|_{Z^1, \bar{\tau}(Z^1)},$$

which Claim 7 will imply is well-defined. Finally, define the tuple

$$(39) \quad \gamma^2 = [\Gamma^1, \Gamma, \text{inc}_{Z^2, X} \circ \delta],$$

which is well-defined since δ maps Z^1 to Z^2 and since $Z^2 \subseteq X$. Figure D.1 depicts the node transformations [1] $\text{inc}_{Z^1, X}$ from γ^1 in (37), [2] $\text{inc}_{Z^2, X} \circ \delta$ from γ^2 in (39), and [3] τ from $[\Theta, \Theta', \tau]$.

Claim 1: (a) $\Theta^1 = (Z^1, E^1, \mathcal{H}^1, \lambda^1)$ is a CLT. (b) $\mathcal{Z}^1 = \{Z^1\}$. By inspection, (Z^1, E^1) is an out-tree whose only run is the walk Z^1 . This establishes (C1) of Definition 2.1 and part (b) of this Claim 1. Thus it remains to show (C2)–(C4). (C2) follows immediately from the definition of \mathcal{H}^1 below (36). For (C3), recall Z^1 is a walk, which implies that each element of Z^1 has exactly one immediate successor, which vacuously implies λ^1 is locally injective. Further, λ^1 is surjective by its definition below (36). Finally, (C4) holds vacuously because \mathcal{H}^1 generates the discrete topology.

Claim 2: (a) $I^1 = W^1$. (b) Γ^1 is a game. For (a), I^1 is by general definition equal to the codomain of κ^1 , which by the definition $\kappa^1 = \text{id}_{W^1}$ below (36) is equal to W^1 . For (b), it suffices to show that the tuple satisfies (G1)–(G3) of Definition 3.1. (G1) holds by Claim 1. For (G2), $\kappa^1 = \text{id}_{W^1}$ is surjective by inspection and continuous because \mathcal{H}^1 generates the discrete topology for W^1 . Finally for (G3), U^1 by its definition below (36) has the form $\langle U_i^1: \{Z^1\} \rightarrow \bar{\mathbb{R}} \rangle_{i \in W^1}$, which by Claim 1(b) is $\langle U_i^1: \mathcal{Z} \rightarrow \bar{\mathbb{R}} \rangle_{i \in W^1}$, which by part (a) is $\langle U_i^1: \mathcal{Z} \rightarrow \bar{\mathbb{R}} \rangle_{i \in I^1}$.

Claim 3: $[\Theta^1, \Theta, \text{inc}_{Z^1, X}]$ is a CLT morphism. Θ^1 is a CLT by Claim 1(a) and Θ is a CLT by assumption. Thus it suffices to show that the tuple satisfies (cE), (cH), and (cL) of Definition 2.4. For (cE), E^1 's definition (35) implies that every edge in E^1 is also in E . Finally, (cH) and (cL) hold vacuously because \mathcal{H}^1 generates the discrete topology.

Claim 4: $\gamma^1 = [\Gamma^1, \Gamma, \text{inc}_{Z^1, X}]$, from definition (37), is a game morphism. Γ^1 is a game by Claim 2(b) and Γ is a game by assumption. Thus it suffices to show (gZ), (gK), and (gU) of Definition 3.4.

For (gZ), Claim 3 shows $[\Theta^1, \Theta, \text{inc}_{Z^1, X}]$ is a morphism. Further, end preservation (18) holds if

$$\overline{\text{inc}_{Z^1, X}}(Z^1 \setminus W^1) \subseteq X \setminus W.$$

To show this, recall Z^1 is a walk. If Z^1 is infinite, $Z^1 \setminus W^1$ is empty. If Z^1 is finite, the only element of $Z^1 \setminus W^1$ is the last node of the walk Z^1 . Since $Z^1 \in \mathcal{Z}$ by Z^1 's definition, that last node is in $X \setminus W$ by \mathcal{Z} 's definition (4).

For (gK), control preservation (20) holds if

$$(\forall x_A \in W^1, x_B \in W^1) \kappa^1(x_A) = \kappa^1(x_B) \text{ implies } \kappa(\text{inc}_{Z^1, X}(x_A)) = \kappa(\text{inc}_{Z^1, X}(x_B)).$$

Since $\kappa^1 = \text{id}_{W^1}$ by its definition below (36), $\kappa^1(x_A) = \kappa^1(x_B)$ implies $x_A = x_B$.

Finally, (gU) holds if

$$(\forall i \in W^1) \zeta^1 \text{ is monotone from } (Z^1, \lesssim_i^1) \text{ to } (\mathcal{Z}, \lesssim_{\iota^1(i)}),$$

where $\langle \lesssim_i^1 \rangle_{i \in W^1}$ is derived from Γ^1 , and where ζ^1 and ι^1 are derived from γ^1 . This holds vacuously because Claim 1(b) shows \mathcal{Z}^1 is the singleton $\{Z^1\}$.

Claim 5: (a) $\tau|_{Z^1, \bar{\tau}(Z^1)}$ is a graph isomorphism (note 33) from (Z^1, E^1) to $(\bar{\tau}(Z^1), E' \cap \bar{\tau}(Z^1)^2)$. (b) $\tau|_{Z^2, \bar{\tau}(Z^2)}$ is a graph isomorphism from (Z^2, E^2) to $(\bar{\tau}(Z^2), E' \cap \bar{\tau}(Z^2)^2)$. Part (a) follows from Lemma D.4 at $N = Z^1$. Part (b) follows from Lemma D.4 at $N = Z^2$.

Claim 6: $\bar{\tau}(Z^1) = \bar{\tau}(Z^2)$. Equation (34b) and ζ 's definition (19) imply

$$(40) \quad P' \circ \tau(r) \cup \bar{\tau}(Z^1) = P' \circ \tau(r) \cup \bar{\tau}(Z^2).$$

Since $(\forall x \in X) r \preceq x$, Proposition 2.5(a) implies $(\forall x \in X) \tau(r) \preceq' \tau(x)$, which implies $P' \circ \tau(r)$ and $\bar{\tau}(X)$ are disjoint. So $Z^1 \subseteq X$ implies $P' \circ \tau(r)$ and $\bar{\tau}(Z^1)$ are disjoint, and $Z^2 \subseteq X$ implies $P' \circ \tau(r)$ and $\bar{\tau}(Z^2)$ are disjoint. So (40) implies the claim.

Claim 7: $\delta = (\tau|_{Z^2, \bar{\tau}(Z^2)})^{-1} \circ \tau|_{Z^1, \bar{\tau}(Z^1)}$, from definition (38), is a graph isomorphism from (Z^1, E^1) to (Z^2, E^2) . Claims 5(b) and 6 imply that $\tau|_{Z^2, \bar{\tau}(Z^2)}$ is a graph isomorphism from (Z^2, E^2) to $(\bar{\tau}(Z^1), E' \cap \bar{\tau}(Z^1)^2)$, so its inverse is a graph morphism from $(\bar{\tau}(Z^1), E' \cap \bar{\tau}(Z^1)^2)$ to (Z^2, E^2) . The result then follows from Claim 5(a).

Claim 8: $[\Theta^1, \Theta, \text{inc}_{Z^2, X} \circ \delta]$ is a CLT morphism. Θ^1 is a CLT by Claim 1, Θ is a CLT by assumption, and $\text{inc}_{Z^2, X} \circ \delta$ maps from Z^1 , the node set of Θ^1 , to X , the node set of Θ , by Claim 7. Thus it suffices to show (cE), (cH), and (cL). For (cE), take $xy \in E^1$. Then Claim 7 implies $\delta(x)\delta(y) \in E^2$, which by E^2 's definition (35) implies $\delta(x)\delta(y) \in E$. Finally, (cH) and (cL) hold vacuously because \mathcal{H}^1 (36) generates the discrete topology.

Claim 9: $\gamma^2 = [\Gamma^1, \Gamma, \text{inc}_{Z^2, X} \circ \delta]$, from definition (39), is a game morphism. Γ^1 is a game by Claim 2(b), Γ is a game by assumption, and $\text{inc}_{Z^2, X} \circ \delta$ maps from Z^1 , the node set of Γ^1 , to X , the node set of Γ , by Claim 7. Thus it suffices to show (gZ), (gK), and (gU).

For (gZ), Claim 8 shows $[\Theta^1, \Theta, \text{inc}_{Z^2, X} \circ \delta]$ is a morphism. Further, it is end preservation (18) holds if

$$\overline{\text{inc}_{Z^2, X} \circ \delta}(Z^1 \setminus W^1) \subseteq X \setminus W.$$

To show this, recall Z^1 is a walk. If Z^1 is infinite, $Z^1 \setminus W^1$ is empty. If Z^1 is finite, the only element of $Z^1 \setminus W^1$ is the last node of the walk Z^1 . So by Claim 7, the only element of $\bar{\delta}(Z^1 \setminus W^2)$ is the last node of the walk Z^2 . Since $Z^2 \in \mathcal{Z}$ by assumption, that node is in $X \setminus W$ by \mathcal{Z} 's definition (4).

For (gK), control preservation (20) holds if

$$(\forall x_A \in W^1, x_B \in W^1) \kappa^1(x_A) = \kappa^1(x_B) \text{ implies } \kappa(\text{inc}_{Z^2, X} \circ \delta(x_A)) = \kappa(\text{inc}_{Z^2, X} \circ \delta(x_B)).$$

Since $\kappa^1 = \text{id}_{W^1}$ by its definition below (36), $\kappa^1(x_A) = \kappa^1(x_B)$ implies $x_A = x_B$.

Finally, (gU) holds if

$$(\forall i \in W^1) \zeta^2 \text{ is monotone from } (Z^1, \lesssim_i^1) \text{ to } (Z, \lesssim_{\iota^2(i)}),$$

where $\langle \lesssim_i^1 \rangle_{i \in W^1}$ is derived from Γ^1 , and where ζ^2 and ι^2 are derived from γ^2 . This holds vacuously because Claim 1(b) shows Z^1 is the singleton $\{Z^1\}$.

Claim 10: $\gamma^1 \neq \gamma^2$. It suffices to show $\text{inc}_{Z^1, X} \neq \text{inc}_{Z^2, X} \circ \delta$. Thus it suffices to show there is a node $x \in Z^1$ such that $x \neq \delta(x)$. Toward that end, we know from (34a) that $Z^1 \setminus Z^2$ or $Z^2 \setminus Z^1$ is nonempty. In the first contingency, take $x \in Z^1 \setminus Z^2$. Then $x \in Z^1$ and Claim 7 imply $\delta(x) \in Z^2$, which by $x \notin Z^2$ implies $\delta(x) \neq x$. In the second contingency, take $x \in Z^2 \setminus Z^1$. First, $x \in Z^2$ and Claim 7 imply there is [A] $x^1 \in Z^1$ such that [B] $\delta(x^1) = x$. Second, $x \notin Z^1$ and [B] imply $\delta(x^1) \notin Z^1$, which by [A] implies $\delta(x^1) \neq x^1$.

Claim 11: $\gamma \circ \gamma^1 = \gamma \circ \gamma^2$. Since $\gamma^1 = [\Gamma^1, \Gamma, \text{inc}_{Z^1, X}]$ and $\gamma^2 = [\Gamma^1, \Gamma, \text{inc}_{Z^2, X} \circ \delta]$ are morphisms by Claims 4 and 9, and since $\gamma = [\Gamma, \Gamma', \tau]$ is a morphism by assumption, it suffices to show $\tau \circ \text{inc}_{Z^1, X} = \tau \circ \text{inc}_{Z^2, X} \circ \delta$. By δ 's definition (38), this is equivalent to

$$\tau \circ \text{inc}_{Z^1, X} = \tau \circ \text{inc}_{Z^2, X} \circ (\tau|_{Z^2, \bar{\tau}(Z^2)})^{-1} \circ \tau|_{Z^1, \bar{\tau}(Z^1)},$$

which by Claim 6 is equivalent to

$$\tau \circ \text{inc}_{Z^1, X} = \tau \circ \text{inc}_{Z^2, X} \circ (\tau|_{Z^2, \bar{\tau}(Z^1)})^{-1} \circ \tau|_{Z^1, \bar{\tau}(Z^1)},$$

which by Claim 5(a) is equivalent to

$$\tau \circ \text{inc}_{Z^1, X} \circ (\tau|_{Z^1, \bar{\tau}(Z^1)})^{-1} = \tau \circ \text{inc}_{Z^2, X} \circ (\tau|_{Z^2, \bar{\tau}(Z^1)})^{-1},$$

which is equivalent to $(\forall x \in \bar{\tau}(Z^1)) \tau \circ \tau^{-1}(x) = \tau \circ \tau^{-1}(x)$.

Conclusion. Claims 10 and 11 imply that γ is not monic. \square

Proof D.6 (for Proposition 4.2). Lemma D.3 shows that the injectivity of ζ is sufficient. Lemma D.5 shows it is necessary. \square

APPENDIX E. FOR ISOMORPHISMS

Lemma E.1. *Suppose that (X, E) and (X', E') are out-trees, that $\tau: X \rightarrow X'$ is a bijection, and that $\{\tau(x)\tau(y) \mid xy \in E\} \subseteq E'$. Then the following hold.*

- (a) $E \ni xy \mapsto \tau(x)\tau(y) \in E'$ is a bijection.
- (b) $\tau|_{W, W'}$ is a bijection.
- (c) $\tau|_{X \setminus W, X' \setminus W'}$ is a bijection.
- (d) $(\forall x \in X, y \in X) x \preceq y \Leftrightarrow \tau(x) \preceq' \tau(y)$.
- (e) $(\forall x \in X, y \in X) x \prec y \Leftrightarrow \tau(x) \prec' \tau(y)$.
- (f) $\tau(r) = r'$.
- (g) $(\forall y \in X) \bar{\tau}(P(y)) = P'(\tau(y))$.
- (h) $\bar{\tau}|_{Z, Z'}$ is a bijection.

Proof. Part (a) immediately implies parts (b)–(h). (Part (a) is equivalent to the out-trees (X, E) and (X', E') being graph-isomorphic.)

To prove part (a), let $E^\tau = \{\tau(x)\tau(y) \mid xy \in E\}$, and note $E^\tau \subseteq E'$. Since τ is injective, $E \ni xy \mapsto \tau(x)\tau(y) \in E^\tau$ is a bijection. Thus it suffices to show $E^\tau = E'$.

Since (X, E) is an out-tree and $\tau: X \rightarrow X'$ is a bijection, (X, E^τ) is an out-tree with root $\tau(r)$. We can show

$$(41) \quad \tau(r) = r',$$

where r' is the root of the out-tree (X', E') . To see this, suppose $\tau(r) \neq r'$. Then there would be a nontrivial walk in (X', E') from r' to $\tau(r)$. Further, there would be a nontrivial walk in (X', E^τ) from $\tau(r)$ to r' , which by $E^\tau \subseteq E'$ implies there is a nontrivial walk in (X', E') from $\tau(r)$ to r' . The concatenation of these two walks would be a nontrivial walk in (X', E') from r' to itself, which contradicts (X', E') being an out-tree.

Since $E^\tau \subseteq E'$, it suffices to show $E' \setminus E^\tau = \emptyset$. Toward that end, suppose

$$(42) \quad x'y' \in E' \setminus E^\tau.$$

Since (X', E^τ) is an out-tree, there are walks in (X', E^τ) from $\tau(r)$ to x' , and from $\tau(r)$ to y' . Thus, since $E^\tau \subseteq E'$ and since $\tau(r) = r'$ by (41), these may also be seen as walks in (X', E') from r' to x' , and from r' to y' . Since $x'y' \in E'$ by (42), the first walk can be augmented into a walk in (X', E') from r' to y' , and this augmented first walk cannot equal the second walk since $x'y' \notin E^\tau$ by (42) and since the second walk is entirely in (X', E^τ) by definition. Thus there are two distinct walks in (X', E') from r' to y' , in contradiction to (X', E') being an out-tree. \square

Lemma E.2. *Suppose Θ and Θ' are CLTs and $\tau: X \rightarrow X'$ is a bijection. Then the following are equivalent.*

- (a) $\tau|_{W, W'}$ is a homeomorphism.
- (b) $(\forall H \in \mathcal{H}) \bar{\tau}(H) \in \mathcal{H}'$.
- (c) $\bar{\tau}|_{\mathcal{H}, \mathcal{H}'}$ is a bijection.

Proof. $(a \Rightarrow b)$. Assume (a) and take an information set $H \in \mathcal{H}$. Note $\tau|_{W, W'}$ being a bijection implies $\bar{\tau}(W) \subseteq W'$, which implies the assumptions of Proposition 2.2 are met. Thus the forward direction of Proposition 2.2 implies there is $H' \in \mathcal{H}'$ such that $\bar{\tau}(H) \subseteq H'$, which by the bijectivity of τ implies

$$(43) \quad H \subseteq \bar{\tau}^{-1}(H').$$

Conversely, $\tau|_{W, W'}$ being a bijection implies $\bar{\tau}^{-1}(W') \subseteq W$, which implies the assumptions of Proposition 2.2 for $[\Theta', \Theta, \tau^{-1}]$ are met. Thus, since $H' \in \mathcal{H}'$, the forward direction of Proposition 2.2 at $[\Theta', \Theta, \tau^{-1}]$ implies there is $H^* \in \mathcal{H}$ such that

$$(44) \quad \bar{\tau}^{-1}(H') \subseteq H^*.$$

But since H and H^* are cells in the partition \mathcal{H} by (cH) of Definition 2.1, equations (43) and (44) imply $H = H^*$ and $H = \bar{\tau}^{-1}(H')$. The latter implies $\bar{\tau}(H) = H'$, and H' belongs to \mathcal{H}' by construction.

$(a \Leftarrow b)$. Assume (b). This implies $\bar{\tau}(W) \subseteq W'$ because \mathcal{H} partitions W by (cH) for Θ , and because \mathcal{H}' partitions W' by (cH) for Θ' . Thus the assumptions of Proposition 2.2 hold. Hence that proposition's reverse direction implies (b).

$(b \Rightarrow c)$. This follows from [1] the bijectivity of τ , [2] that \mathcal{H} partitions W by (cH) for Θ , and that \mathcal{H}' partitions W' by (cH) for Θ' .

$(b \Leftarrow c)$. Part (c) implies that $\bar{\tau}$ takes each member of \mathcal{H} to a member of \mathcal{H}' . This is part (b). \square

Lemma E.3. *Suppose $[\Theta, \Theta', \tau]$ is a morphism and τ is bijective. Then the following hold.*

- (a) $(\forall x \in W) \alpha_x$ is bijective.
- (b) the action transformation of the tuple $[\Theta', \Theta, \tau^{-1}]$ is $\langle \alpha_{\tau^{-1}(x')}^{-1} \rangle_{x' \in W'}$.
- (c) $[\Theta, \Theta', \tau]$ is end-preserving.
- (d) $\zeta = \bar{\tau}|_{Z, Z'}$.

Proof. (a). Take a decision node $x \in W$. By definition (11), $\alpha_x: F(x) \rightarrow F'(\tau(x))$. To show its injectivity, take feasible actions $a_1 \in F(x)$ and $a_2 \in F(x)$. It suffices to show that $\alpha_x(a_1) = \alpha_x(a_2)$ implies $a_1 = a_2$. Toward that end, suppose $\alpha_x(a_1) = \alpha_x(a_2)$.

Then α_x 's definition (11) implies

$$\lambda'(\tau(x)\tau(n(x, a_1))) = \lambda'(\tau(x)\tau(n(x, a_2))).$$

Thus λ' 's local injectivity (C3) implies $\tau(n(x, a_1)) = \tau(n(x, a_2))$. Thus the injectivity of τ implies $n(x, a_1) = n(x, a_2)$. Thus Lemma B.1(c) at $x_o = x$ implies $a_1 = a_2$.

It remains to show that α_x is surjective. Since $\alpha_x: F(x) \rightarrow F'(\tau(x))$ by definition (11), it suffices to show $\overline{\alpha_x}(F(x)) \supseteq F'(\tau(x))$. Toward that end, take $a' \in F'(\tau(x))$. By the definition of F' , there is $y' \in X'$ such that $a' = \lambda'(\tau(x)y')$. Thus by the bijectivity of τ , there is $y \in X$ such that $a' = \lambda'(\tau(x)\tau(y))$. Thus Proposition 2.5(f) implies [1] $a' = \alpha_x(\lambda(xy))$. Meanwhile, the definition of F implies $\lambda(xy) \in F(x)$. So [1] implies $a' \in \overline{\alpha_x}(F(x))$.

(b). By definition (11), it suffices to show

$$(\forall x' \in W', a' \in F'(x')) \alpha_{\tau^{-1}(x')}^{-1}(a') = \lambda(\tau^{-1}(x')\tau^{-1}(n'(x', a'))).$$

Thus, since $\tau|_{W, W'}$ is a bijection by Lemma E.1(b), it suffices to show

$$(\forall x \in W, a' \in F'(\tau(x))) \alpha_x^{-1}(a') = \lambda(x\tau^{-1}(n'(\tau(x), a'))).$$

Thus, since each $\alpha_x: F(x) \rightarrow F'(\tau(x))$ is a bijection by part (a), it suffices to show

$$(\forall x \in W, a \in F(x)) a = \lambda(x\tau^{-1}(n'(\tau(x), \alpha_x(a)))).$$

Toward that end, take a decision node $x \in W$ and feasible action $a \in F(x)$. In steps, a by Lemma B.1(c) is equal to $\lambda(xn(x, a))$, which by the bijectivity of τ is equal to $\lambda(x\tau^{-1}(\tau(n(x, a))))$, which by Proposition 2.5(g) is equal to $\lambda(x\tau^{-1}(n'(\tau(x), \alpha_x(a))))$.

(c). Since $X \setminus W$ consists of the end nodes of Θ , and since $X' \setminus W'$ consists of the end nodes of Θ' , Lemma E.1(c) implies that $[\Theta, \Theta', \tau]$ is end-preserving.

(d). Recall from definition (19) that $\zeta: \mathcal{Z} \rightarrow \mathcal{Z}'$ is defined by $\zeta(Z) = P' \circ \tau(r) \cup \bar{\tau}(Z)$. Lemma E.1(f) implies $P' \circ \tau(r) = \emptyset$. Thus $\zeta = \bar{\tau}|_{\mathcal{Z}, \mathcal{Z}'}$. \square

Lemma E.4. *Suppose $[\Theta, \Theta', \tau]$ is a morphism such that τ is a bijection and $\tau|_{W, W'}$ is a homeomorphism. Then $[\Theta, \Theta', \tau]$ is an isomorphism.*

Proof. It suffices to show that $[\Theta', \Theta, \tau^{-1}]$ is a morphism. Thus by Definition 2.4, it suffices to show that $[\Theta', \Theta, \tau^{-1}]$ satisfies (cE), (cH), and (cL). First, (cE) for $[\Theta', \Theta, \tau^{-1}]$ is $(\forall x' y' \in E') \tau^{-1}(x')\tau^{-1}(y') \in E$, which follows from Lemma E.1(a). Second, (cH) for $[\Theta', \Theta, \tau^{-1}]$ is the continuity of $\tau^{-1}|_{W', W}$, which follows from $\tau|_{W, W'}$ being a homeomorphism.

Finally, by Lemma E.3(b), (cL) for $[\Theta', \Theta, \tau^{-1}]$ is equivalent to the continuity of $\langle \alpha_{\tau^{-1}(x')}^{-1} \rangle_{x' \in W'}$ from W' . Since $\tau|_{W, W'}$ is a homeomorphism, this is equivalent to the continuity of $\langle \alpha_x^{-1} \rangle_{x \in W}$ from W , which is equivalent to the continuity of $\langle \alpha_x \rangle_{x \in W}$ from W , which is (cL) for $[\Theta, \Theta', \tau]$. \square

Proof E.5 (for Proposition 4.3). Lemma E.4 shows that $[\Theta, \Theta', \tau]$ is an isomorphism whenever it satisfies (a) and (b). So it remains to show that whenever $[\Theta, \Theta', \tau]$ is an isomorphism, it satisfies (a)–(p). Toward that end, suppose $[\Theta, \Theta', \tau]$ is an isomorphism.

(a) and the conclusions of Lemmas E.1–E.3. Since $[\Theta, \Theta', \tau]$ is an isomorphism, τ is a bijection, which is condition (a). Further, since $[\Theta, \Theta', \tau]$ is a morphism, both (X, E) and (X', E') are out-trees. Yet further, since $[\Theta, \Theta', \tau]$ is a morphism, it satisfies (cE) of Definition 2.4, which is edge-preservation (10), which is equivalent to $\{ \tau(x)\tau(y) \mid xy \in E \} \subseteq E'$. Thus all the assumptions of Lemmas E.1–E.3 are met, and hence all their conclusions follow.

(b,m). Since $[\Theta, \Theta', \tau]$ is an isomorphism, its inverse is $[\Theta', \Theta, \tau^{-1}]$, which is condition (m). Further, we have three facts: [1] $\tau|_{W, W'}$ is a bijection by Lemma E.1(b), [2] $\tau|_{W, W'}$ is continuous by (cH) for $[\Theta, \Theta', \tau]$, and [3] $(\tau|_{W, W'})^{-1}$ is continuous by (cH) for $[\Theta', \Theta, \tau^{-1}]$. Thus $\tau|_{W, W'}$ is a homeomorphism, which is condition (b).

(c–h). Lemma E.1(c) implies (c). Lemma E.1(a) implies (d). Finally, Lemma E.1(d)–(g) implies (e)–(h).

(i). This follows from part (b) and Lemma E.2(a \Rightarrow c).

(j–l). Lemma E.3(a) implies (j). Lemma E.3(c) implies (k). Finally, Lemma E.3(d) implies $\zeta = \bar{\tau}|_{Z, Z'}$, which by Lemma E.1(h) is a bijection. Thus (l) holds.

(n–p). Lemma E.3(b) implies (n). Lemma E.3(c) applied to $[\Theta', \Theta, \tau^{-1}]$ implies (o). Finally, Lemma E.3(d) applied to $[\Theta', \Theta, \tau^{-1}]$ implies $\zeta' = \overline{\tau^{-1}}|_{Z', Z}$, which by part (l) is ζ^{-1} . Thus (p) holds. \square

Lemma E.6. *Let $[G, G', \tau]$ be a tuple such that G and G' are games and $[\Theta, \Theta', \tau]$ is a morphism. Also let ι^* be a function. Then the following are equivalent.*

(a) $[G, G', \tau]$ is control-preserving (20) and ι^* is its player transformation (21).

(b) $\iota^* \circ \kappa = \kappa' \circ \tau|_{W, W'}$.

Proof. (a \Rightarrow b). Proposition 3.3(b).

(a \Leftarrow b). To show $[G, G', \tau]$ is control-preserving (20), it must be shown that $(\forall x_1 \in W, x_2 \in W) \kappa(x_1) = \kappa(x_2)$ implies $\kappa(\tau(x_1)) = \kappa(\tau(x_2))$. Toward that end, take two decision nodes $x_1 \in W$ and $x_2 \in W$ such that $\kappa(x_1) = \kappa(x_2)$. Then $\iota^* \circ \kappa(x_1) = \iota^* \circ \kappa(x_2)$, which by (b) implies $\kappa' \circ \tau(x_1) = \kappa' \circ \tau(x_2)$.

To show ι^* is the player transformation (21) of $[G, G', \tau]$, it must be shown that [1] $\iota^*: I \rightarrow I'$ and that [2] $(\iota^*)^{\mathbf{gr}} = \{ (\kappa(x), \kappa'(\tau(x))) \mid x \in W \}$. For [1], note that ι^* is a function by assumption, that its domain by (b) is κ 's codomain which is I , and that its codomain by (b) is κ' 's codomain which is I' . Finally, [2] follows immediately from (b). \square

Lemma E.7. *Suppose $[G, G', \tau]$ is a morphism such that (a) $[\Theta, \Theta', \tau]$ is an isomorphism, (b) ι is bijective, and (c) $(\forall i \in I) \zeta$ is a preorder isomorphism from (Z, \lesssim_i) to $(Z', \lesssim'_{\iota(i)})$. Then $[G, G', \tau]$ is an isomorphism.*

Proof. Since $[\Theta, \Theta', \tau]$ is an isomorphism, we have all the conclusions of Proposition 4.3. Part (a) of the proposition shows τ is a bijection, so τ^{-1} exists, so the tuple $[G', G, \tau^{-1}]$ exists. Thus it suffices for this lemma to show that $[G', G, \tau^{-1}]$ is a morphism. In other words, it suffices to show that $[G', G, \tau^{-1}]$ satisfies conditions (gZ), (gK), and (gU) of Definition 3.4.

First, (gZ) requires that $[\Theta', \Theta, \tau^{-1}]$ be an end-preserving morphism. This follows from Proposition 4.3(o).

Second, (gK) requires that $[Γ', Γ, τ^{-1}]$ be control-preserving (20). To show this and that $ι^{-1}$ is the player transformation of $[Γ', Γ, τ^{-1}]$, note Proposition 3.3(b) implies $ι ∘ κ = κ' ∘ τ|_{W, W'}$. Since $τ|_{W, W'}$ is a bijection by Proposition 4.3(b), this is equivalent to $ι ∘ κ ∘ τ^{-1}|_{W', W} = κ'$. Since $ι$ is a bijection by assumption, this is equivalent to $κ ∘ τ^{-1}|_{W', W} = ι^{-1} ∘ κ'$. This is equivalent to Lemma E.6 condition (b) for the tuple $[Γ', Γ, τ^{-1}]$ and for $ι^* = ι^{-1}$. Thus Lemma E.6(b⇒a) implies that $[Γ', Γ, τ^{-1}]$ is control-preserving, and that $ι^{-1}$ is its player transformation.

Third, consider (gU). Since $ζ^{-1}$ is the run transformation of $[Γ', Γ, τ^{-1}]$ by Proposition 4.3(p), and since $ι^{-1}$ is the player transformation of $[Γ', Γ, τ^{-1}]$ by the previous paragraph, (gU) requires that

$$(∀i' ∈ I') ζ^{-1} \text{ is monotone from } (\mathcal{Z}', \lesssim_{i'}) \text{ to } (\mathcal{Z}, \lesssim_{ι^{-1}(i')}).$$

Since $ι: I → I'$ is a bijection by assumption, this is equivalent to

$$(∀i ∈ I) ζ^{-1} \text{ is monotone from } (\mathcal{Z}', \lesssim_{ι(i)}) \text{ to } (\mathcal{Z}, \lesssim_i).$$

This holds because the lemma assumes that $(∀i ∈ I) ζ$ is a preorder isomorphism from $(\mathcal{Z}, \lesssim_i)$ to $(\mathcal{Z}', \lesssim_{ι(i)})$. \square

Proof E.8 (for Proposition 4.4). Lemma E.7 shows that $[Γ, Γ', τ]$ is an isomorphism whenever it satisfies (a), (b), and (c). So it remains to show that whenever $[Γ, Γ', τ]$ is an isomorphism, it satisfies (a)–(e). Toward that end, suppose $[Γ, Γ', τ]$ is an isomorphism.

(a, d). Since $[Γ, Γ', τ]$ is an isomorphism, and since there is a forgetful functor from **Gm** to **CLT** by Proposition 3.6, $[Θ, Θ', τ]$ is a **CLT** isomorphism. This is condition (a). Further, since $[Γ, Γ', τ]$ is an isomorphism, $[Γ', Γ, τ^{-1}]$ is its inverse. This is condition (d).

(b, e). Let $ι^*$ be the player transformation of $[Γ', Γ, τ^{-1}]$. It suffices for both (b) and (e) to show that $ι^* = ι^{-1}$. Since $[Θ', Θ, τ^{-1}] ∘ [Θ, Θ', τ] = \text{id}_Θ$, Proposition 3.5(b) and Lemma C.3 imply [1] $ι^* ∘ ι = \text{id}_I$. Similarly, since $[Θ, Θ', τ] ∘ [Θ', Θ, τ^{-1}] = \text{id}_{Θ'}$, Proposition 3.5(b) and Lemma C.3 imply [2] $ι ∘ ι^* = \text{id}_{I'}$. Since $ι: I → I'$ and $ι^*: I' → I$ by definition (21), results [1] and [2] imply $ι^* = ι^{-1}$.

(c). Since $[Γ, Γ', τ]$ is a morphism, condition (gU) of Definition 3.4 implies

$$(45) \quad (∀i ∈ I) ζ \text{ is monotone from } (\mathcal{Z}, \lesssim_i) \text{ to } (\mathcal{Z}', \lesssim_{ι(i)}).$$

By part (a) and Proposition 4.3(p), the run transformation of $[Γ', Γ, τ^{-1}]$ is $ζ^{-1}$. By part (e), the player transformation of $[Γ', Γ, τ^{-1}]$ is $ι^{-1}$. Therefore, since $[Γ', Γ, τ^{-1}]$ is a morphism, condition (gU) of Definition 3.4 implies

$$(∀i' ∈ I') ζ^{-1} \text{ is monotone from } (\mathcal{Z}', \lesssim_{i'}) \text{ to } (\mathcal{Z}, \lesssim_{ι^{-1}(i')}).$$

Since $ι: I → I'$ is a bijection by part (b), this is equivalent to

$$(46) \quad (∀i ∈ I) ζ^{-1} \text{ is monotone from } (\mathcal{Z}', \lesssim_{ι(i)}) \text{ to } (\mathcal{Z}, \lesssim_i).$$

The combination of (45) and (46) is equivalent to

$$(∀i ∈ I) ζ \text{ is preorder isomorphism } (\mathcal{Z}, \lesssim_i) \text{ to } (\mathcal{Z}', \lesssim_{ι(i)}).$$

\square

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