Higher-order circuits

Matt Wilson[1]

[1]Université Paris-Saclay, CentraleSupélec, Inria, CNRS, LMF, 91190 Gif-sur-Yvette, France

Based on the preprint of the same title available at arXiv:2602.18701.

In the past two decades of quantum information theory a simple concept has been studied in considerable depth: the notion of *circuit-theoretic hole* and the *higher-order operations* that might be applied to it. Depending on the perspective, such a hole or operation might represent: the most general kind of manipulation which can be applied to a computational gate [1], the most general spacetime environment which can be probed by agent-interventions [2], a node within a causal structure [3], a generalisation of the notion of non-markovian process to the quantum regime [4], or even a quantum game [5].

It is not surprising to find the higher-order concept to be ubiquitous. A function of a function, a process on a process, is no big stretch of the imagination, particularly in the light of the lambda calculus and its proposed quantum generalisations [6]. More broadly however, as outlined above, the higher-order view appears to be a recurring theme - appearing whenever aspects of the interaction between agent and environment (particularly for the purpose of handling causal influence) need to be quantised. Despite this, an open problem remains in our understanding of the concept of circuit-theoretic holes, and so in our understanding of causal modelling in general theories beyond finite dimensional quantum mechanics. That problem, is on the completion of a somewhat deceivingly simple to state analogy:

$$\begin{aligned} \text{Circuits} \quad &\leftrightarrow \quad \text{Monoidal Categories,} \\ \text{(Higher-order) Circuits} \quad &\leftrightarrow \quad \text{(Higher-order) Monoidal Categories.} \end{aligned}$$

In other words, we do not have an algebraic model for what it means to be a higher-order circuit theory.

A natural proposal for higher-order monoidal categories, is to use the already established categorical notion of *closed* monoidal categories [7]. Closure in a monoidal category captures a salient feature of higher-order functions, namely, that they can be curried to give a natural isomorphism of the form $\mathcal{C}(a \otimes b, c) \cong \mathcal{C}(a, [b, c])$. This is a structure with a notion of space, and a notion of higher order morphisms as those of type $[a, a'] \to [b, b']$ in $\mathcal{C}$. However, this view adds something unnecessary and misses something necessary to the study of circuit-theoretic holes:

- The existence of iterated higher-order structure in closed monoidal categories requires iterated notions of higher-order morphism, which appear unnecessary for the qualification of a theory as a theory of holes. Indeed, in the theory of quantum supermaps, the majority of attention is given to those higher-order maps which are no-more than 2^{nd}-order [1, 2].
- A key aspect of holes as modelled by quantum supermaps is that one need not insert an entire process into a hole, instead one might only insert part of a multipartite process into a hole. There is nothing in the structure of a closed monoidal category which allows for this. One cannot for instance impose that:

  $$\mathcal{C}(A, A') \otimes \mathcal{C}(B, B') \cong \mathcal{C}(A \otimes B, A' \otimes B').$$

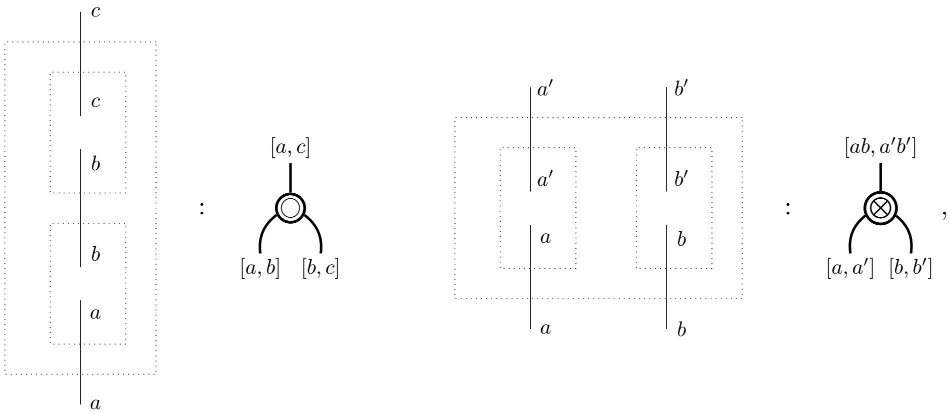
  Indeed, to ask for such an isomorphism as argued in [8], is close to request the existence of pathological time loops.

A resolution to the first point [9] put forward an interpretation of higher-order maps in terms of enriched categories [10], and a resolution to the second point [8] put forward that the base for enrichment should be a symmetric polycategory [11]. This proposal to study higher-order circuits as monoidal categories enriched in symmetric polycategories is quite distinct from the traditional closed-monoidal view on what it means to be higher-order. It is certainly broad enough that it avoids imposing unnecessary structure onto theories of higher-order maps, however, it is natural to ask whether it might be still too broad to capture everything that is salient regarding higher-order processes.

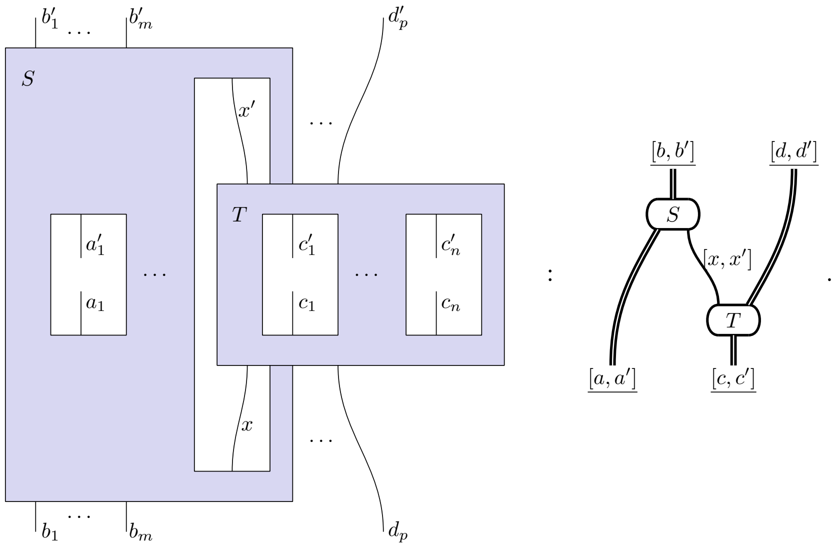
In this article, we build an abstract model for the notion of higher-order circuit theories based on enrichment into polycategories by further adding a minimal additional structure (a frobenius-like cotensor) which formalises the existence of parallel pairs of holes. We consider the direct consequences of our higher-order circuit model, in particular, proving that any higher-order circuit theory embeds into the theory of strong profunctors [12, 13, 14], an abstract model put forward for the generalisation of higher-order transformations (and so indefinite causal structures) to infinite-dimensional and operational probabilistic theories [15, 16].

Enrichment and multi-arity composition laws: We first give an outline description for the salient features of a symmetric monoidal category $\mathcal{C}$ enriched into a symmetric polycategory $\mathcal{P}$, focussing on giving intuition for the higher-order concepts that they correspond to. Such an algebraic structure then, consists of:

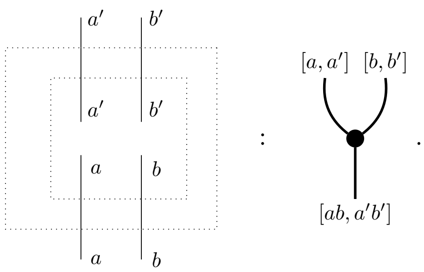
- A collection of objects $a, b, c, \ldots$
- For each pair of objects a, b an associated object in $\mathcal{P}$ denoted $[a, b]$ (also, a notion of identity state is included on $[a, a]$ for each a),
- Families of *sequential composition* and *parallel composition* morphisms (we write each morphism formally on the right and write it's informal interpretation in terms of higher-order operations on the left):



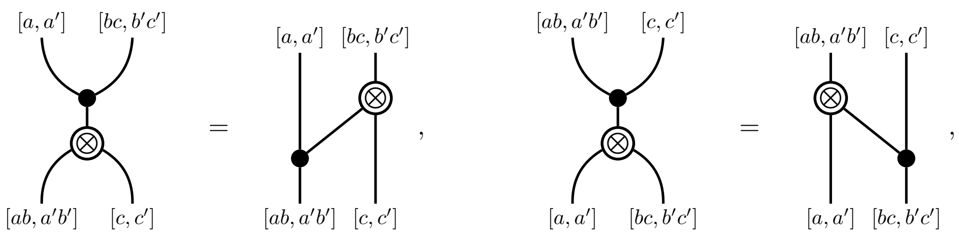
The polycategory structure allows us to introduce partial-nesting, this is *the* key idea in both the linear-algebraic [17] and categorical [15] approaches to defining quantum supermaps:



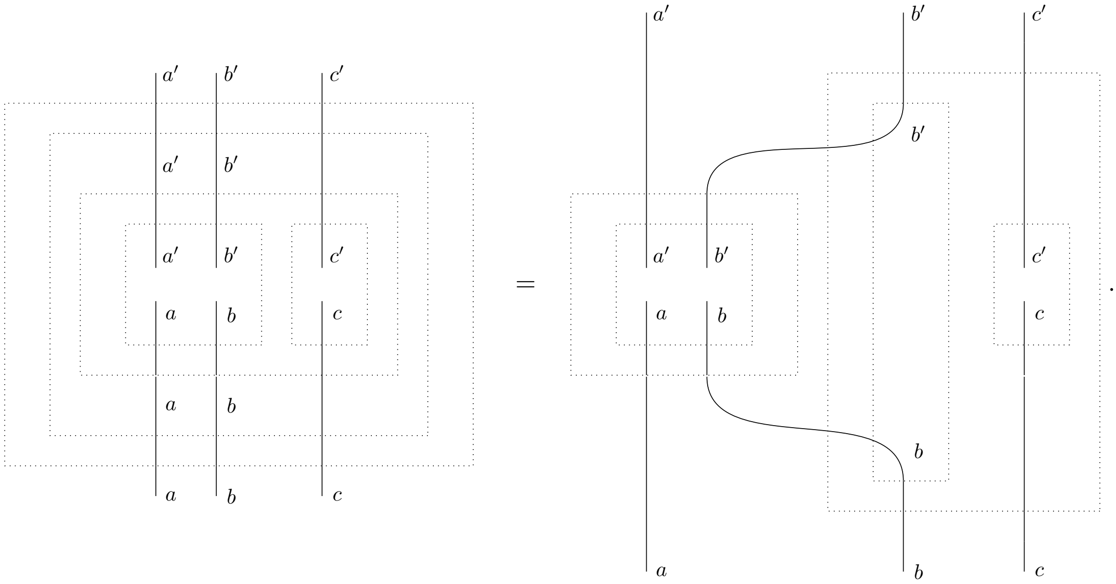
Inspired by the Yoneda lemma for polycategories [18], we use the cotensor to represent parallel wire-gaps



We impose additional laws which encode coherence between the co-tensor and enrichment structure. As a concrete example of a higher-order circuit coherence, there are the *frobenius laws* given by:



which model the relationships between ways of arranging and composing wire-gaps. The former law for instance, represents equivalence between two formal interpretations of three-parallel-gaps:



Examples of higher-order circuit theories which somehow make use of compact closed categories in their construction include: (trivially) compact closed categories [19], (more interestingly) higher-order causal categories $\textbf{Caus}[\mathcal{C}]$ constructed from pre-causal categories $\mathcal{C}$ [20], the $\mathcal{D}$-supermaps on $\mathcal{C}$ for any embedding of a symmetric monoidal category $\textbf{C}$ into a compact closed category $\mathcal{D}$ [21, 8]. Examples of higher-order circuit theories which only make use of symmetric monoidal structure (and so apply to all operational probabilistic theories and infinite dimensional quantum theory) include polyslots [8], and single-party representable supermaps [8].

Upper Bound for Higher-Order Circuits: We find an upper bound on the possibilities for higher-order circuits over a fixed base theory $\mathcal{C}$ of lower order processes. The upper bound is given by the theory of strong profunctors [12, 13, 14], which is used to give a general construction for the bare minimum requirements of supermaps [15].

Theorem. *For every higher-order circuit theory* $(\mathcal{P}, \mathcal{C})$ *there exists a faithful multifunctor*

$$\mathcal{F} : \mathcal{P}^{\#} \to \textbf{StrProf}[\mathcal{C}],$$

where $\mathcal{P}^{\#}$ *is the quotient of* $\mathcal{P}$ *by behavioural equivalence on states.*

Since it has been proven that on the symmetric monoidal category of quantum channels the morphisms between strong profunctors model the quantum superchannels [21, 15], this demonstrates that on the quantum channels there is no higher-order circuit theory broader than higher-order quantum theory as it is traditionally conceptualised [22, 20].

Outlook and impact: Higher-order circuits, albeit in the case in which all possible compositional structures are treated as strict, can be given a relativity simple categorical semantics in terms of polycategories, enrichment, and cotensors. Each law for higher-order circuits has an interpretation as a graphically tautology, and with only those laws in place it is possible to find an upper bound on higher-order circuits based on strong profunctors. As a result, we appear to have distilled an intuitive concept into categorical algebra. A variety of questions and potential applications naturally follow:

- How can the definition of higher-order circuits be adapted to the more complicated case in which $\mathcal{C}$ is no-longer forced to be strict. In particular, could there be strictification and coherence theorems [23] for higher-order circuits suitably categorified?
- Can the planar diagrams with holes be used to build a sound and complete graphical language for higher-order circuits, in the same way that string diagrams are sound and complete for monoidal categories [24]? Could this be combined with the ZX-calculus to give sound and complete calculi within higher-order quantum theory?
- Higher-order quantum operations [22, 20] can be equipped with rather complex composition operations [25, 26, 27, 16]. Most notably in the introduction of a sequencing tensor product which gives higher-order operations the structure of BV-categories in the representable case [28]. Can the notion of a sequencing structure be added to the polycategorical setting, or, is it already handled by enrichment?
- Given that we have distilled what it means to be a higher-order circuit theory, can we give a general approach to higher-order resource theories, by defining them as (sub) higher-order circuits in analogy with [29] which defines lower-order resource theories as (sub) circuit theories.

More broadly, this works aims to contribute towards development of a stable theory-independent framework for holes, higher-order processes, and causal influence through the agent-environment interaction.

1 Acknowledgements

The majority of this work was completed while the author was a PhD student at the Quantum Group of the Computer Science Department at the University of Oxford, and can be found in the associated thesis [30]. Refinements have been made since, both while affiliated to University College London and now CentraleSupelec, University of Paris-Saclay. The author (MW) is particularly grateful to Giulio Chiribella (GC) and James Hefford (JH), this work puts together many ideas produced through ongoing conversations and projects in collaboration with both GC and JH. From GC in particular, comes the idea that multi-input higher-order processes could be modelled using theories with preferred sequential and parallel composition *morphisms* [9]; and from JH comes the idea that strong profunctors could provide the domain for locally-applicable transformations and so generalise the spaces on which they act [15]. MW is also grateful to Peter Selinger, and Benoit Valiron, for useful conversations during the development of this manuscript. MW was funded by the Engineering and Physical Sciences Research Council grant number EP/L015242/1 while at the University of Oxford and grant number EP/W524335/1 while at University College London. This project/publication was made possible through the support of the ID#62312 grant from the John Templeton Foundation, as part of "The Quantum Information Structure of Spacetime" Project (QISS). The opinions expressed in this project/publication are those of the author(s) and do not necessarily reflect the views of the John Templeton Foundation. This work has also been funded by the French National Research Agency (ANR), (i) by the project TaQC ANR-22-CE47-0012, and (ii) within the framework of "Plan France 2030", under the research projects EPIQ ANR-22-PETQ-0007, OQULUS ANR-23-PETQ-0013, HQI-Acquisition ANR-22-PNCQ-0001 and HQI-R&D ANR-22-PNCQ-0002.

References

[1] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Transforming quantum operations: Quantum supermaps," *EPL (Europhysics Letters)*, vol. 83, no. 3, p. 30004, 7 2008.

[2] O. Oreshkov, F. Costa, and Ĉ. Brukner, "Quantum correlations with no causal order," *Nature Communications*, vol. 3, 2012.

[3] J. Barrett, R. Lorenz, and O. Oreshkov, "Cyclic quantum causal models," *Nature Communications*, vol. 12, no. 1, p. 885, 2021. [Online]. Available: https://doi.org/10.1038/s41467-020-20456-x

[4] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, "Non-Markovian quantum processes: complete framework and efficient characterisation," *Physical Review A*, vol. 97, no. 1, p. 012127, Jan 2018.

[5] G. Gutoski and J. Watrous, "Toward a general theory of quantum games," in *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, ser. STOC07. ACM, Jun. 2007, pp. 565–574. [Online]. Available: http://dx.doi.org/10.1145/1250790.1250873

[6] P. Selinger and B. Valiron, *A Lambda Calculus for Quantum Computation with Classical Control.* Springer Berlin Heidelberg, 2005, pp. 354–368. [Online]. Available: http://dx.doi.org/10.1007/11417170_26

[7] S. Eilenberg and G. M. Kelly, "Closed Categories," pp. 421–562, 1966. [Online]. Available: https://link.springer.com/chapter/10.1007/978-3-642-99902-4_22

[8] M. Wilson and G. Chiribella, "Free polycategories for unitary supermaps of arbitrary dimension," 2022.

[9] ——, "A mathematical framework for transformations of physical processes," 2023.

[10] P. T. Johnstone, "BASIC CONCEPTS OF ENRICHED CATEGORY THEORY (London Mathematical Society Lecture Note Series, 64)," *Bulletin of the London Mathematical Society*, vol. 15, no. 1, pp. 96–96, 1 1983. [Online]. Available: http://doi.wiley.com/10.1112/blms/15.1.96

[11] M. Szabo, "Polycategories," *Communications in Algebra*, vol. 3, no. 8, pp. 663–689, 1975. [Online]. Available: https://doi.org/10.1080/00927877508822067

[12] R. Paré and L. Román, "Dinatural numbers," *Journal of Pure and Applied Algebra*, vol. 128, no. 1, pp. 33–92, 1998. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0022404997000364

[13] D. Tambara, "Distributors on a tensor category," *Hokkaido Mathematical Journal*, vol. 35, no. 2, pp. 379–425, May 2006.

[14] C. Pastro and R. Street, "Doubles for monoidal categories," 2007. [Online]. Available: https://arxiv.org/abs/0711.1859

[15] J. Hefford and M. Wilson, "A Profunctorial Semantics for Quantum Supermaps," in *Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science*, ser. LICS '24. New York, NY, USA: Association for Computing Machinery, 2024.

[16] ——, "A bv-category of spacetime interventions," 2025. [Online]. Available: https://arxiv.org/abs/2502.19022

[17] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Transforming quantum operations: Quantum supermaps," *EPL (Europhysics Letters)*, vol. 83, no. 3, p. 30004, 2008.

[18] M. Shulman, "*-autonomous envelopes and conservativity," *Electronic Proceedings in Theoretical Computer Science*, vol. 353, pp. 175–194, Dec. 2021. [Online]. Available: http://dx.doi.org/10.4204/EPTCS.353.9

[19] G. Kelly and M. Laplaza, "Coherence for compact closed categories," *Journal of Pure and Applied Algebra*, vol. 19, pp. 193–213, 1980. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0022404980901012

[20] A. Kissinger and S. Uijlen, "A categorical semantics for causal structure," *Logical Methods in Computer Science*, vol. 15, 2019.

[21] M. Wilson, G. Chiribella, and A. Kissinger, "Quantum supermaps are characterized by locality," 2025. [Online]. Available: https://arxiv.org/abs/2205.09844

[22] A. Bisio and P. Perinotti, "Theoretical framework for higher-order quantum theory," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 475, no. 2225, p. 20180706, 2019. [Online]. Available: http://dx.doi.org/10.1098/rspa.2018.0706

[23] S. M. Lane, *Categories for the Working Mathematician*, ser. Graduate Texts in Mathematics. Springer New York, 1971, vol. 5, 10.1007/978-1-4612-9839-7.

[24] A. Joyal and R. Street, "The geometry of tensor calculus, I," *Advances in Mathematics*, vol. 88, no. 1, pp. 55–112, 7 1991.

[25] L. Apadula, A. Bisio, and P. Perinotti, "No-signalling constrains quantum computation with indefinite causal structure," *Quantum*, vol. 8, p. 1241, Feb. 2024. [Online]. Available: https://doi.org/10.22331/q-2024-02-05-1241

[26] W. Simmons and A. Kissinger, "A complete logic for causal consistency," 2024.

[27] ——, "Higher-Order Causal Theories Are Models of BV-Logic," in *47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022)*, ser. Leibniz International Proceedings in Informatics (LIPIcs), S. Szeider, R. Ganian, and A. Silva, Eds., vol. 241. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, pp. 80:1–80:14.

[28] R. Blute, P. Panangaden, and S. Slavnov, "Deep inference and probabilistic coherence spaces," *Applied Categorical Structures*, vol. 20, pp. 209–228, 2012.

[29] B. Coecke, T. Fritz, and R. W. Spekkens, "A mathematical theory of resources," *Inf. Comput.*, vol. 250, pp. 59–86, 2016.

[30] M. Wilson, "Compositional frameworks for supermaps and causality," PhD thesis, University of Oxford, 2023.