A Category-theoretic Reconstruction of Logical Expressivism

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In the philosophical tradition of 'analytic pragmatism', which attempts to account for linguistic meanings in terms of their practices of use, logical expressivism is a theory which offers a distinct perspective on logic. We shed light on Brandom and Hlobil's recent formalization of logical expressivism by reconstructing its formal semantics through the use of universal constructions. This reveals similarities with categorical logic: there is a focus on internalizing judgment structure, and connectives arise from adjunctions.

1 Background: inferentialism and logical expressivism

In this section, we briefly summarize some concepts from *Reasons for Logic, Logic for Reasons* [1]. We can understand a domain of reasoning as having a collection of claims (that which can be asserted in that domain's vocabulary) and a notion of consequence, i.e. of some claims following from other claims. Example domains could range from 'group theory' to 'the game of chess' to '18th century maritime law'. A special class of examples comes from formal logics (e.g. classical, linear, paraconsistent) and their consequence relations. These consequence relations, despite their diversity, share some common structure. For example, the consequence *operator*, which takes a set of premises to its set of consequences, is often assumed to satisfy the Tarskian conditions of reflexivity, monotonicity, and idempotence. From the proof-theoretic perspective, such constraints respectively correspond to structural rules of identity, weakening, and cut. The formalization of a domain, i.e. the encoding of its claims within a formal system such that its consequence relation coincides with the logical consequence relation, is a powerful tool if successfully accomplished. This is because weakening allows for portability of conclusions in different contexts, and cut allows for composability of reasoning. Thus it is a remarkable achievement that we have suitable logical encodings for some domains: for example, quantum logic for statements about quantum-mechanical observables [2], intuitionistic logic for constructive mathematics, linear logic for resource-sensitive reasoning.

Inspired by the success of formalization of (certain aspects of) scientific reasoning, one could take the view that good reasoning always has this form: insofar as we genuinely mean something by 'It is a dog' and 'It is a mammal', and the former is a reason for the latter, then it must be the case that there must exist *some* encoding in *some* logic that is able to recover this inference as a logical consequence. Russell's *On Denoting* [3] offers an example of logical analysis of ordinary language expressions like "the present King of France" using classical predicate logic; however, one can reject his analysis on the basis of it *not* reproducing our ordinary inferences: e.g. he would be forced to endorse "The present King of France is bald" is a good reason for "The moon is made of cheese", as his analysis of the antecedent is that it is meaningful but false. There are many ways to react to this failure: one could seek a different encoding in classical logic, one can seek to develop new kinds of logics (e.g. free logics) to deal with 'non-existent entities', and, lastly, we emphasize one could reject the attempt to ground the domain of interest in logical reasoning. It may be the case that legal, ethical, and medical reasoning do not satisfy

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the Tarskian constraints.[1]

In contrast to this logico-semantic approach to meaning, *inferentialism* about meaning is the thesis that we should firstly think of the meaning of a sentence in terms of its good inferences and secondarily (if at all) think of it as represented by a formal term in some deductive calculus or an element in some semantic domain; this is the opposite order of explanation relative to the story explained above [6, 7]. We *begin* with some norm governing good judgments involving atomic sentences.

Definition 1.1 (Signed incompatibility frames)**.** A *signed incompatibility frame* is a set X of propositional atoms, equipped with a subset $\bot \subseteq \mathbb{N}[X+X]$. *Idempotent* signed incompatibility frames are sets X equipped with $\bot \subseteq \mathcal{P}[X+X]$. Elements of $\mathbb{N}[X+X]$ (or $\mathcal{P}[X+X]$ in the idempotent case) are called *positions*, and $\bot$ is the subset of *incompatible* positions [1, Def. 62].

We equivalently refer to a signed incompatibility frame as an *implication frame* because we often interpret a position (Γ,Δ) as a candidate implication $\Gamma \vdash \Delta$, with sets (or multisets) of atoms on each side of the turnstile. Such a candidate implication may or may not obtain, according to the frame; the 'good implications', i.e. the ones which obtain, comprise $\bot$.[2] For example, $a,b \vdash a,c$ is notation for $(\{a,b\},\{a,c\}) \in \bot$ and $a,b \nvdash c$ is notation for $(\{a,b\},\{c\}) \notin \bot$. Idempotent frames implicitly enforce contraction as a structural rule, and both frames implicitly enforce exchange.

Thinking of an implication frame's turnstile as 'raw, input data' leads to *logical expressivism*, which takes the concept of inferences being 'simply' good (as in an implication frame) to be explanatorily-prior to the concept of logically-good inferences [9]. This can be formalized via the sequent rules of non-monotonic, multi-succedent logic (NMMS, [10, 11]), seen in Fig. 1, which derive the goodness of propositions built from logical connectives. Adding logical connectives expands a vocabulary from X to $\mathrm{BF}(X)$, the set of Boolean formulas on propositional atoms X. Because NMMS rules are bidirectional, any sequent which involves logically-complex propositions is tantamount to (a conjunction of) statements involving fewer logical connectives and, ultimately, to a assertion that some particular set of sequents is contained within $\bot$. We can understand the new, logically complex propositions as being descriptions of $\bot$. This is the idea of logical expressivism: that the characteristic function of logic is to express features of some antecedent, prelogical system of implications. Here 'express' means to make explicit, i.e. internalize, features of $\bot$ that, were originally only describable in a metavocabulary.[3]

Definition 1.2 (Logical elaboration)**.** *Logical elaboration* is the mapping, sending implication frames to implication frames, which sends some $(X,\bot \subseteq \mathbb{N}[X+X])$ to $(\mathrm{BF}(X),\bot' \subseteq \mathbb{N}[\mathrm{BF}(X)+\mathrm{BF}(X)])$, with $\bot'$ characterized via the noncontractive rules of Fig. 1. We can also logically elaborate idempotent implication frames $(X,\bot \subseteq \mathcal{P}[X+X])$, sending them to $(\mathrm{BF}(X),\bot' \subseteq \mathcal{P}[\mathrm{BF}(X)+\mathrm{BF}(X)])$ using the contractive rules of Fig. 1 for deriving $\bot'$.

The 'implication space semantics' is defined relative to a choice of an implication frame, $(X,\bot)$. At its core is an operation on sets of candidate implications.

[1] Of the three constraints characteristic of logical consequence, weakening is the easiest to drop. Categorical logic, by moving from cartesian closed categories to symmetric monoidal closed categories, allows for viewing weakening as a property rather than assumed structure; however, identity and cut remain fixed. We note that rejection of cut is one way to make sense of logical paradoxes (such as the liar sentence) [4], and there many ordinary language (non-paradoxical) examples [5].

[2] Following [8], a *bilateralist* reading of the turnstile $\Gamma \vdash \Delta$ is that 'It is normatively out-of-bounds' to *accept* everything in Γ and *reject* everything in Δ. Interpreting sequents in this way (where the left hand side represents accepted claims, and the right hand side represents simultaneously rejected claims) explains why an 'incompatible' position (i.e. out-of-bounds) is a 'good' implication (Δ follows if you cannot reject it while accepting Γ).

[3] In [11, Ex. 1], it is shown that, in a radically substructural setting, the $\otimes$L and $\otimes$R rules of linear logic are *not* expressive in this sense: one can derive $p\otimes q \vdash p\otimes q$ from atomic sequents in two different ways, such that its assertion does not determinately tell us something about $\bot \subseteq \mathbb{N}[X+X]$. It can be derived starting from $(p \vdash p)$ and $(q \vdash q)$ *or* starting from $(p,q \vdash q)$ and $(\dashv q)$.

$$\frac{\Gamma, a \vdash \Delta}{\Gamma \vdash \neg a, \Delta}\neg R \quad \frac{\Gamma \vdash a, \Delta}{\Gamma, \neg a \vdash \Delta}\neg L \quad \frac{\Gamma \vdash a, \Delta \quad \Gamma \vdash b, \Delta}{\Gamma \vdash a \wedge b, \Delta}\wedge R \quad \frac{\Gamma, a, b \vdash \Delta}{\Gamma, a \wedge b \vdash \Delta}\wedge L \quad \frac{\Gamma \vdash a, b, \Delta}{\Gamma \vdash a \vee b, \Delta}\vee R$$

$$\frac{\Gamma, a \vdash \Delta \quad \Gamma, b \vdash \Delta}{\Gamma, a \vee b \vdash \Delta}\vee L \quad \frac{\Gamma \vdash a, \Delta \quad \Gamma \vdash b, \Delta \quad \Gamma \vdash a, b, \Delta}{\Gamma \vdash a \wedge b, \Delta}\wedge R^{c} \quad \frac{\Gamma, a \vdash \Delta \quad \Gamma, b \vdash \Delta \quad \Gamma, a, b \vdash \Delta}{\Gamma, a \vee b \vdash \Delta}\vee L^{c}$$

Figure 1: Sequent rules for NMMS, with contractive (demarcated by superscript ^{c}) and non-contractive variants for $\wedge$R and $\vee$L rules. All other rules are considered both contractive and non-contractive.

Definition 1.3 (Range of subjunctive robustness)**.** Given an implication frame $(X, \perp)$, the *range of subjunctive robustness* (RSR: $\mathcal{P}[\mathbb{N}[X+X]] \to \mathcal{P}[\mathbb{N}[X+X]]$ or $\mathcal{P}[\mathcal{P}[X+X]] \to \mathcal{P}[\mathcal{P}[X+X]]$ in the idempotent setting) function sends a set of candidate implications $\{(\Gamma_1, \Delta_1), ..., (\Gamma_i, \Delta_i)\}$ to the set $\{(\Theta, \Omega) \mid \forall i\colon (\Gamma_i \cup \Theta, \Delta_i \cup \Omega) \in \perp\}$ [1, Def. 63]. We will abbreviate RSR(A) as A^*.

RSR sends an individual sequent to the set of contexts in which it can be added to make a good implication (hence, $(\{(\varnothing, \varnothing)\})^* = \perp$). We then define implicational roles $\mathbb{R} := \text{im(RSR)}$ and the set of semantic values (called "conceptual contents" [1, Def. 66]) as $\mathbb{C} := \mathbb{R}^2$. Roles can be combined via 'symjunction' (defined as $A \sqcap B := (A \cup B)^{**}$ [1, Def. 69]) and 'adjunction'[4] (defined as $A \sqcup B := (\{(\Gamma \cup \Gamma', \Delta \cup \Delta') \mid (\Gamma, \Delta) \in A,\ (\Gamma', \Delta') \in B\})^{**}$ [1, Def. 67]). There is also a semantic consequence relation: letting $\vec{\mathtt{A}}, \vec{\mathtt{B}} \subseteq \mathbb{C}$, we have $\vec{\mathtt{A}} \vDash \vec{\mathtt{B}} := (\bigsqcup_{\langle \mathtt{a}_+,\, \mathtt{a}_- \rangle \in \vec{\mathtt{A}}} \mathtt{a}_+) \sqcup (\bigsqcup_{\langle \mathtt{b}_+,\, \mathtt{b}_- \rangle \in \vec{\mathtt{B}}} \mathtt{b}_-) \subseteq \perp$ [1, Def. 68].These operations are also used in semantic clauses for Boolean expressions (and, in the noncontractive case, linear logic expressions). Letting $[\![A]\!] := \langle \mathtt{a}_+,\ \mathtt{a}_- \rangle$ and $[\![B]\!] := \langle \mathtt{b}_+,\ \mathtt{b}_- \rangle$, the following definitions are semantic clauses for classical and linear logic as defined in [1, Defs. 70, 88].

$$[\![\neg A]\!] := \langle \mathtt{a}_-, \mathtt{a}_+ \rangle \qquad [\![A \wedge B]\!] := \langle \mathtt{a}_+ \sqcup \mathtt{b}_+,\ \mathtt{a}_- \sqcap \mathtt{b}_- \sqcap (\mathtt{a}_- \sqcup \mathtt{b}_-) \rangle$$
$$[\![A \vee B]\!] := [\![\neg(\neg A \wedge \neg B)]\!] \qquad [\![A \to B]\!] := [\![\neg A \vee B]\!]$$
$$[\![A \otimes B]\!] := \langle \mathtt{a}_+ \sqcup \mathtt{b}_+,\ (\mathtt{a}_-^* \sqcup \mathtt{b}_-^*)^* \rangle \qquad [\![A \oplus B]\!] := \langle \mathtt{a}_+ \sqcap \mathtt{b}_+,\ (\mathtt{a}_-^* \sqcap \mathtt{b}_-^*)^* \rangle$$
$$[\![A \parr B]\!] := [\![\neg(\neg A \otimes \neg B)]\!] \qquad [\![A \& B]\!] := [\![\neg(\neg A \oplus \neg B)]\!]$$

In addition to satisfying our desideratum of applying logic without imposing idempotency or monotonicity constraints on our consequence operation, these choices of semantic values and clauses are shown to have many noteworthy logical properties: for example, we can understand a correspondence between many propositional logic formalisms and conditions on the starting implication frame. However, despite strong philosophical motivations, there are many mathematical choices above which may seem unusual or non-obvious to a traditional logician. We aim to show that these choices are natural and can be reconstructed via universal constructions, with the implication space semantics arising as the unit of an adjunction. Fig. 2 gives a visual overview of the structure of the paper.

Related work: In [12], Corfield applies the logical expressivist interpretation of introduction and elimination rules to constructions in dependent type theory. In [13], Rosenthal constructs Girard quantales from phase spaces and shows every Girard quantale can be obtained in this way. In [14], the authors explore relationships between ordered commutative monoids and Girard quantales, although they do not go as far as defining a category of phase spaces. The generation of a *-autonomous category from a symmetric monoidal category with a distinguished object to serve as the dualizing object bears a *prima facie* connection to the Chu construction [15]; however, this construction fails to be relevant to reconstructing logical expressivism because the monoidal product is quite different from what is needed. Lastly, there

[4] An unfortunate collision in terminology.

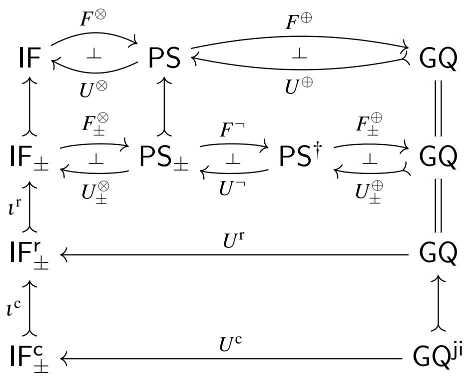


Figure 2: Structure of paper: the top row is covered in Section 2, which describes IF, a category of single-sided consequence relations, and adjunctions $F^{\otimes} \dashv U^{\otimes}$ (whose unit adds new elements corresponding to conjunctions) and $F^{\oplus} \dashv U^{\oplus}$ (which adds disjunctions). The second row is covered in Section 3, which concerns $\mathsf{IF}_{\pm}$, a category of two-sided consequence relations. The adjunctions respectively add conjunction, negation, and disjunction. This section also features a restriction of the overall adjunction to reflexive relations ($\mathsf{IF}^{\mathsf{r}}_{\pm}$), drawn in the third row. Section 4 concerns the last row, restricting the overall adjunction to relations which enforce contraction and containment ($\mathsf{IF}^{\mathsf{c}}_{\pm}$).

are key differences between the present approach and traditional categorical logic that prevent a simple characterization of how they are related. In the categorical logic following [16], propositions are objects of a category and derivations $\Gamma \vdash A$ are morphisms. In contrast, our setting is proof-irrelevant, and whether sequents $\Gamma \vdash A$ obtain or not is simply data which may (but need not) satisfy properties such as transitivity that would be automatic if derivations were morphisms in a category.

Notation: We denote multisets of elements from some set X with $\mathbb{N}[X]$, whereas the power set of X is $\mathcal{P}[X]$. We liberally apply the bijections $\mathbb{N}[X+X] \cong \mathbb{N}[X] \times \mathbb{N}[X]$ and $\mathcal{P}[X+X] \cong \mathcal{P}[X] \times \mathcal{P}[X]$ implicitly. Functor compositions like GF are shorthand for $G \circ F$, though functors (and morphisms in general) are sometimes composed in diagrammatic order, written $F \cdot G$. At times, pairs will be written $\langle -,- \rangle$ rather than $(-,-)$, especially if the elements inside contain further pairs. All mentioned monoids and quantales will be commutative and unital. Lastly, we warn that $\perp$ and $(-)^{\perp}$ are overloaded notations which have distinct but related meanings in the categories we will encounter.

2 Unsigned logic

2.1 Defining categories of phase spaces and incompatibility frames

We start with phase spaces, a core component of Girard's phase semantics for linear logic.

Definition 2.1 (Phase spaces)**.** A (commutative) *phase space* is a commutative monoid equipped with a distinguished subset. A phase space $\mathcal{X} := (X,+,0,\perp \subseteq X)$ has a natural $(-)^{\perp}$ operation on its elements, $a^{\perp} := \{x \mid x+a \in \perp\}$, as well as on subsets of its elements $A^{\perp} := \bigcap_{a \in A} a^{\perp} = \{x \mid \forall a \in A \colon x+a \in \perp\}$. This also leads to a natural order on its elements: $a \leq_{\mathcal{X}} b := a^{\perp} \supseteq b^{\perp}$.

A Girard quantale is a thin *-autonomous cocomplete category, i.e. a quantale with a dualizing object, $\perp$.[5] Phase semantics naturally associates a particular Girard quantale to any phase space [17].

Definition 2.2 (Natural Girard quantale of a phase space)**.** Let $\mathcal{X} := (X,+,0,\perp \subseteq X)$ be a phase space implicitly ordered by $\leq_{\mathcal{X}}$, and let $\mathcal{X}^{\downarrow}$ be the lattice of lower sets of $\mathcal{X}$. This lattice has a closure operator: $(-)^{\perp\perp}$. We define $\mathrm{Gir}(\mathcal{X})$ to be the natural Girard quantale of $\mathcal{X}$: its elements are $(-)^{\perp\perp}$-closed subsets of X, ordered by subset inclusion, $A \otimes B := \{a+b \mid a \in A,\ b \in B\}^{\perp\perp}$, and with dualizing element $\perp$,

Lemma 2.1. For any phase space $(X,+,0,\perp)$, we have $\perp = \perp^{\perp\perp}$.

[5] Not to be confused with the bottom element of the quantale's lattice.

Note that $\bot$ is an element of $\mathrm{Gir}(\mathcal{X})$ by Lemma 2.1. The closure induces a quotient $q_{\mathcal{X}}\colon \mathcal{X}^{\downarrow} \twoheadrightarrow \mathrm{Gir}(\mathcal{X})$ sending each set to its closure. We also observe the principal lower sets of $\mathcal{X}^{\downarrow}$ are closed ($x^{\downarrow} = x^{\downarrow\bot\bot}$), which entails that $q_{\mathcal{X}}$ is an order embedding for the principal lower sets: $q_{\mathcal{X}}(a^{\downarrow}) \leq_{\mathrm{Gir}(\mathcal{X})} q_{\mathcal{X}}(b^{\downarrow}) \iff a^{\bot\bot} \subseteq b^{\bot\bot} \iff b^{\bot} \subseteq a^{\bot} \iff a \leq_{\mathcal{X}} b$. We now construct a category of phase spaces, starting with a Girard quantales.

Definition 2.3 (Category of Girard quantales)**.** Let GQ have Girard quantales as objects and, for morphisms $\mathcal{X} \to \mathcal{Y}$, quantale homomorphisms which weakly preserve $\bot$, i.e. $f(\bot_{\mathcal{X}}) \leq_{\mathcal{Y}} \bot_{\mathcal{Y}}$.

There is a forgetful functor $U^{\bot}\colon \mathsf{GQ} \to \mathsf{Quant}$ which discards the structure of having a dualizing object. There is also a free functor $F^{\vee}\colon \mathsf{PreOrdCMon} \to \mathsf{Quant}$: it freely adds joins to the underlying preorder and defines $A \otimes B := \{a+b \mid a \in A,\ b \in B\}^{\downarrow}$.[6] The right adjoint $U^{\vee}\colon \mathsf{Quant} \to \mathsf{PreOrdCMon}$ forgets the property of having all joins, and the hom-set bijection of $F^{\vee} \dashv U^{\vee}$ naturally associates to each quantale morphism $\phi\colon F^{\vee}(\mathcal{P}) \to \mathcal{Q}$ a corresponding $\mathsf{PreOrdCMon}$ morphism $\tilde{\phi}\colon \mathcal{P} \to U^{\vee}(\mathcal{Q})$.

Definition 2.4 (Category of phase spaces)**.** Let PS be the full subcategory of the comma category $F^{\vee} \downarrow U^{\bot}$ (with projection functors $\pi_{\mathsf{PreOrdCMon}}$ and π_{GQ}). It is restricted to the objects, i.e. triples $(\mathcal{P},\ \mathcal{Q},\ \phi\colon F^{\vee}(\mathcal{P}) \to U^{\bot}(\mathcal{Q}))$, where ϕ is surjective and $\tilde{\phi}\colon \mathcal{P} \to U^{\vee}U^{\bot}(\mathcal{Q})$ is an order embedding. Let $\iota_{\mathrm{surj}}\colon \mathsf{PS} \rightarrowtail F^{\vee} \downarrow U^{\bot}$ be the subcategory inclusion and define $F^{\oplus} := \iota_{\mathrm{surj}} \cdot \pi_{\mathsf{GQ}}$.

Lemma 2.2. The objects of PS are in bijection with phase spaces (Definition 2.4), where $(X,+,0,\bot \subseteq X)$ is identified with $((X,+,0,\leq_{\mathcal{X}}),\ \mathrm{Gir}(\mathcal{X}),\ q_{\mathcal{X}})$.

Definition 2.4 fixes a notion of morphism of phase spaces. To concretely consider a morphism $(\mathcal{P},\mathcal{Q},\phi) \to (\mathcal{P}',\mathcal{Q}',\phi')$ in PS: this is a preordered monoid morphism $f\colon \mathcal{P} \to \mathcal{P}'$ and a Girard quantale morphism $g\colon \mathcal{Q} \to \mathcal{Q}'$ such that the square on the right commutes:

$$\begin{array}{ccc} F^{\vee}(\mathcal{P}) & \overset{\phi}{\twoheadrightarrow} & \mathcal{Q} \\ {\scriptstyle F^{\vee}(f)}\downarrow & & \downarrow{\scriptstyle g} \\ F^{\vee}(\mathcal{P}') & \underset{\phi'}{\twoheadrightarrow} & \mathcal{Q}' \end{array}$$

Because ϕ and ϕ' are surjective, g is fully determined by f, so we can think of morphisms just as those preordered monoid maps $\mathcal{P} \to \mathcal{P}'$ that induce a quantale morphism $F^{\vee}(\mathcal{P})^{\bot\bot} \to F^{\vee}(\mathcal{P}')^{\bot\bot}$. To express this constraint more concretely, note g is monotone and weakly preserves $\bot$, thus $f(\bot) \subseteq \bot'$. We get another constraint from requiring the square to commute: for every lower set $A \subseteq P$, we need $f(A)^{\bot'\bot'} = f(A^{\bot\bot})^{\bot'\bot'}$, a condition we call *continuity*.

Lemma 2.3. Continuity of a function $f: X \to Y$ between phase spaces $(X,+,0_X,\bot)$ and $(Y,+,0_Y,\bot')$ is equivalently stated as $\forall A,B \subseteq X\colon A^{\bot} \subseteq B^{\bot} \implies f(A)^{\bot'} \subseteq f(B)^{\bot'}$.[7]

Let $U^{\bot}_{+}\colon \mathsf{PS} \to \mathsf{CMon}$ forget $\bot$, and let $F^{+}\colon \mathsf{Set} \to \mathsf{CMon}$ be the free commutative monoid functor.

Definition 2.5 (Category of incompatibility frames)**.** The category $\mathsf{IF} := \mathsf{PS} \times_{\mathsf{CMon}} \mathsf{Set}$ is the pullback of $U^{\bot}_{+}\colon \mathsf{PS} \to \mathsf{CMon}$ and $F^{+}\colon \mathsf{Set} \to \mathsf{CMon}$. We call the objects of IF (unsigned) *incompatibility frames*. Concretely, these are sets X equipped with a subset of multisets $\bot \subseteq \mathbb{N}[X]$. Morphisms are functions which preserve $\bot$ (i.e. $f(\bot_{\mathcal{X}}) \subseteq \bot_{\mathcal{Y}}$) and are continuous functions, i.e. for all $A,B \subseteq \mathbb{N}[X]$ we have $A^{\bot} \subseteq B^{\bot} \implies f(A)^{\bot} \subseteq f(B)^{\bot}$.

The $(-)^{\bot}$ operation of phase spaces can be applied to incompatibility frames: $(\Gamma \in \mathbb{N}[X])^{\bot} := \{\Gamma' \in \mathbb{N}[X] \mid \Gamma' + \Gamma \in \bot\}$. Likewise, $(x \in X)^{\bot} := \{x\}^{\bot}$ and $(\vec{\Gamma} \subseteq \mathbb{N}[X])^{\bot} := \bigcap_{\Gamma \in \vec{\Gamma}} \Gamma^{\bot}$. Overall, the construction of (unsigned) incompatibility frames is summarized in Fig. 3L.

[6] When our preordered commutative monoids are viewed as thin symmetric monoidal categories, this coincides with Day convolution (see [18], where it is denoted by $\mathscr{P}$). One reason we work with $\mathsf{PreOrdCMon}$ rather than $\mathsf{ThinSMC}$ is that the additional involution structure we will add this category (in Definition 3.1) does not admit a natural description as categorical structure on thin SMCs.

[7] In Lemma 2.3 we additionally show this is equivalent to $\forall A \subseteq X\colon f(A^{\bot_{\mathcal{X}}\bot_{\mathcal{X}}}) \subseteq f(A)^{\bot_{\mathcal{Y}}\bot_{\mathcal{Y}}}$. We call this condition *continuity* because, in topological terms, maps for which the image of the closure is contained in the closure of the image are called *continuous* maps [19, Def 16.A.1].

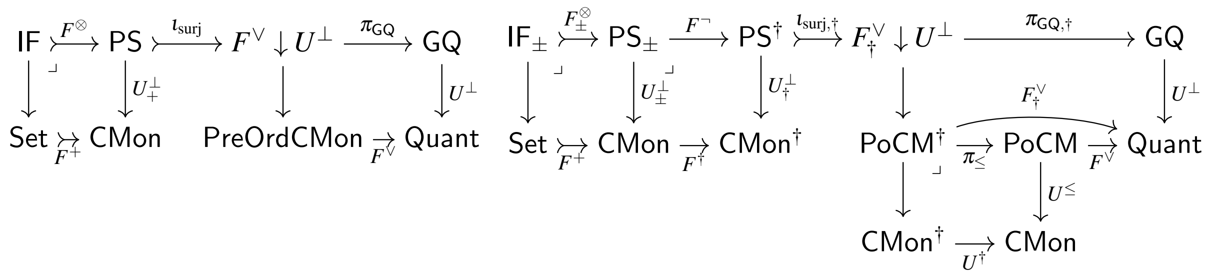
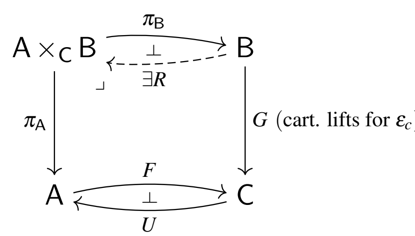


Figure 3: Constructions of IF and $\mathsf{IF}_\pm$ (**Left:** unsigned (Definition 2.5), **Right:** signed (Definition 3.4)).

2.2 Free Girard quantales from incompatibility frames

Notation: We denote the cartesian lift of a morphism f with $\overline{f}$.

Lemma 2.4. Let $F\colon \mathsf{A} \to \mathsf{C}$ and $G\colon \mathsf{B} \to \mathsf{C}$ be functors such that $F \dashv U$ (with unit η and counit ε) and G has cartesian lifts $\overline{\varepsilon_c}$ for all $\varepsilon_c\colon FU(c) \to c$. Then, the pullback projection $\pi_{\mathsf{B}}\colon \mathsf{A} \times_{\mathsf{C}} \mathsf{B} \to \mathsf{B}$ has a right adjoint $R\colon \mathsf{B} \to \mathsf{A} \times_{\mathsf{C}} \mathsf{B}$ given by $R(b) \mapsto (UG(b), \mathrm{dom}(\overline{\varepsilon_{G(b)}}))$.



Lemma 2.5. The forgetful functor $U^\perp_+\colon \mathsf{PS} \to \mathsf{CMon}$ has cartesian lifts for surjections in CMon.

All ε components of $F^+ \dashv U^+$ in CMon are surjections, therefore by Lemmas 2.4 and 2.5 the projection map $F^\otimes\colon \mathsf{IF} \to \mathsf{PS}$ has a right adjoint, $U^\otimes$. This functor $U^\otimes$ sends a phase space $(X,+,0,\perp \subseteq X)$ to the incompatibility frame $(X, \perp' \subseteq \mathbb{N}[X])$, and multisets of X (formal sums of elements in X) are in $\perp'$ iff their actual sum is in $\perp$.

Lemma 2.6. Let $F\colon \mathsf{A} \to \mathsf{C}$ and $G\colon \mathsf{B} \to \mathsf{C}$ be functors with $F \dashv U$. Then the comma category $F \downarrow G$ has a reflective subcategory $R\colon \mathsf{B} \rightarrowtail (\mathsf{F} \downarrow \mathsf{G})$ whose reflector is the projection $\pi_{\mathsf{B}}\colon (F \downarrow G) \to \mathsf{B}$. Concretely, R sends $b \mapsto (UG(b), b, \varepsilon_{G(b)})$ and f to (UGf, f).

Because $F^\vee$ has a right adjoint, by Lemma 2.6 the projection π_{GQ} is the reflector for GQ as a reflective subcategory of $F^\vee \downarrow U^\perp$.

Lemma 2.7. The subcategory functor $\mathsf{GQ} \rightarrowtail \mathsf{F}^\vee \downarrow \mathsf{U}^\perp$ induced by Lemma 2.6 restricts to PS.

By Lemma 2.7, we have a reflective subcategory with $F^\oplus\colon \mathsf{PS} \to \mathsf{GQ}$ as the reflector. $U^\oplus\colon \mathsf{GQ} \rightarrowtail \mathsf{PS}$ sends a Girard quantale $(Q, \otimes, I_Q, \leq, \perp_Q)$ to the phase space $(Q, \otimes, I_Q, \perp_Q^\downarrow)$. We can compose these two adjunctions and obtain an adjunction $F^{\otimes\oplus} \dashv U^{\otimes\oplus}$. Its unit $\eta^{\otimes\oplus}$ sends $x \in (X,\perp)$ to the principal lower set $x^\downarrow$, an element of free Girard quantale of the free phase space of $(X,\perp)$.

2.3 Logical interpretation of unsigned incompatibility frames

Much like a category might not have products, it need not be the case that there exists a function $\otimes\colon \mathbb{N}[X] \to X$ such that $\forall \Gamma \in \mathbb{N}[X] : \otimes(\Gamma)^\perp = \Gamma^\perp$. It also need not be the case there exists a function $\oplus\colon \mathcal{P}(\mathbb{N}[X]) \to X$ such that $\forall \vec{\Gamma} \subseteq \mathbb{N}[X] : \oplus(\vec{\Gamma})^\perp = \vec{\Gamma}^\perp$. A logical expressivist would say that any such functions $\otimes$ and $\oplus$ *make conjunction* (resp. *disjunction*) *explicit* because the binary cases of these constraints are expressed by bidirectional sequent rules, quantified over all $\Gamma \in \mathbb{N}[X]$:

$$\frac{\Gamma, a, b \vdash}{\Gamma, a \otimes b \vdash}\otimes \qquad\qquad \frac{\Gamma, a \vdash \qquad \Gamma, b \vdash}{\Gamma, a \oplus b \vdash}\oplus$$

We interpret the adjunctions $F^{\otimes} \dashv U^{\otimes}$ and $F^{\oplus} \dashv U^{\oplus}$ respectively as the free addition of these structures to an incompatibility frame. One criterion of adequacy for any introduction of logical connectives via sequent rules is conservativity. This criterion states the propriety of judgments which do not feature the newly introduced connectives should be unchanged (the introduction of *tonk* being the paradigmatic counterexample [20]). The judgments which do not feature $\otimes$ are those which are in the image of $\eta^{\otimes}$. We characterize this property in our setting below and note that $\eta^{\otimes}$ is a conservative IF morphism.

Definition 2.6 (Conservative morphisms)**.** A morphism of incompatibility frames $f\colon \mathcal{X} \to \mathcal{Y}$ is *conservative* if $\forall \Gamma \in \mathbb{N}[X]\colon \Gamma \in \bot_{\mathcal{X}} \iff f(\Gamma) \in \bot_{\mathcal{Y}}$.

Proposition 2.1. $\eta^{\otimes} \cdot \eta^{\oplus}$ is conservative.

Proof. Suppose an implication frame $\mathcal{X} := (X, \bot)$ validates $\Gamma \vdash$ for some $\Gamma \in \mathbb{N}[X]$, i.e. $\sum_i \gamma_i \in \bot$. To check conservativity of $\eta^{\otimes\oplus}_{\mathcal{X}}$, we must confirm that $\eta^{\otimes\oplus}(\gamma_1), ..., \eta^{\otimes\oplus}(\gamma_n) \vdash$ in $\hat{\mathcal{X}}$. This becomes a matter of checking if $\bigotimes_i \{\gamma_i\}^{\downarrow} \subseteq \bot$. An arbitrary element of the former set is a position in $\mathcal{X}$ which lies below the sum of a choice from each set $\{\gamma_i\}^{\downarrow}$. The largest elements of these sets are the γ_i themselves, so we can simply check whether $\sum_i \gamma_i$ is an element of $\bot$, which was precisely our starting assumption. □

The story so far is incomplete, as we are only capable of representing single-sided sequents.

3 Signed logic

3.1 Defining categories of involutive phase spaces and implication frames

To recover the two-sidedness of sequents, we begin with involutive commutative monoids.

Lemma 3.1. $U^{\dagger}\colon \mathsf{CMon}^{\dagger} \to \mathsf{CMon}$ forgets the involutive structure of involutive commutative monoids. This is both left and right adjoint to $F^{\dagger}\colon \mathsf{CMon} \to \mathsf{CMon}^{\dagger}$ which sends $\mathcal{Y} := (Y, +, 0)$ to $(Y^2, +^2, 0^2, \sigma)$ and monoid homomorphisms $f\colon X \to Y$ to $f \times f\colon X^2 \to Y^2$. The free unit $\eta^{\dagger}_{\mathcal{X}}\colon \mathcal{X} \to U^{\dagger}F^{\dagger}\mathcal{X}$ in CMon sends $x \mapsto (x, 0)$ and counit $\varepsilon_{\mathcal{Y}}\colon F^{\dagger}U^{\dagger}\mathcal{Y} \to \mathcal{Y}$ sending $(y_1, y_2) \mapsto y_1 + y_2^{\dagger}$. The cofree unit $\eta_{\dagger\mathcal{Y}}\colon \mathcal{Y} \to F^{\dagger}U^{\dagger}\mathcal{Y}$ in $\mathsf{CMon}^{\dagger}$ sends $y \mapsto (y, y^{\dagger})$ and counit $\varepsilon_{\dagger,\mathcal{X}}\colon U^{\dagger}F^{\dagger}\mathcal{X} \to \mathcal{X}$ sending $(x_1, x_2) \mapsto x_1$.

We will need $U^{\leq}\colon \mathsf{PreOrdCMon} \to \mathsf{CMon}$, which discards $\leq$ from preordered commutative monoids.

Definition 3.1 (Involutive preordered monoids)**.** Let $\mathsf{PreOrdCMon}^{\dagger} \cong \mathsf{CMon}^{\dagger} \times_{\mathsf{CMon}} \mathsf{PreOrdCMon}$ be the category of involutive preordered commutative monoids, defined as the pullback of $U^{\dagger}$ and $U^{\leq}$. Concretely, the objects are ordered, involutive commutative monoids, but no constraints are imposed on the relations between $\dagger$ and $\leq$. Morphisms preserve $+$, $\leq$, and $\dagger$.

Let $\pi_{\leq}\colon \mathsf{PreOrdCMon}^{\dagger} \to \mathsf{PreOrdCMon}$ be a projection map of this pullback, and let $F^{\vee}_{\dagger} := \pi_{\leq} \cdot F^{\vee}$.

Definition 3.2 (Category of involutive phase spaces)**.** The category $\mathsf{PS}^{\dagger}$ is full subcategory of $F^{\vee}_{\dagger} \downarrow U^{\perp}$, restricted to surjective ϕ with $\tilde{\phi}$ being an order embedding. Let $\iota_{\mathsf{surj},\dagger}\colon \mathsf{PS}^{\dagger} \rightarrowtail F^{\vee}_{\dagger} \downarrow U^{\perp}$ be the inclusion, and let $F^{\oplus}_{\pm}$ be $\iota_{\mathsf{surj},\dagger} \cdot \pi_{\mathsf{GQ},\dagger}$

Moving from PS to $\mathsf{PS}^{\dagger}$ equips phase spaces with an involution bearing no particular relation to $\bot$, and morphisms must preserve this involution. There is a forgetful functor $U^{\perp}_{\dagger}\colon \mathsf{PS}^{\dagger} \to \mathsf{CMon}^{\dagger}$.

Definition 3.3 (Category of signed phase spaces)**.** The category $\mathsf{PS}_{\pm} \cong \mathsf{PS}^{\dagger} \times_{\mathsf{CMon}^{\dagger}} \mathsf{CMon}$ of signed phase spaces is the pullback of $U^{\perp}_{\dagger}$ and $F^{\dagger}$. Concretely, its objects are commutative monoids with a subset $\bot \subseteq X^2$. For any $A \subseteq X^2$, $A^{\perp} := \{(p_+, p_-) \in X^2 \mid \forall (a_+, a_-) \in A\colon (p_+ + a_+,\ p_- + a_-) \in \bot\}$. and its morphisms are functions $f\colon X \to Y$ such that $(f \times f)(\bot_{\mathcal{X}}) \subseteq \bot_{\mathcal{Y}}$ and are continuous, i.e. for all $A, B \subseteq X^2$ we have $A^{\perp} \subseteq B^{\perp} \implies f(A)^{\perp'} \subseteq f(B)^{\perp'}$.

There is a forgetful functor $U^{\perp}_{\pm}\colon \mathsf{PS}_{\pm} \to \mathsf{CMon}$ discarding the $\perp$ information.

Definition 3.4 (Category of signed incompatibility frames)**.** The category $\mathsf{IF}_{\pm} \cong \mathsf{PS}_{\pm} \times_{\mathsf{CMon}} \mathsf{Set}$ of involutive incompatibility frames is the pullback of $U^{\perp}_{\pm}$ and F^{+}. Concretely, its objects are the implication frames of Definition 1.1, and its morphisms are continuous functions $f\colon X \to Y$ such that $(f \times f)(\perp_{\mathcal{X}}) \subseteq \perp_{\mathcal{Y}}$.

The overall construction of $\mathsf{IF}_{\pm}$ is summarized in Fig. 3R. Although general implication frames are very expressive in their capacity to represent unrestricted consequence relations, we will have special interest in ones which are reflexive.

Definition 3.5 (Reflexive frames)**.** An element $x \in X$ of an incompatibility frame $(X, \perp \subseteq \mathbb{N}[X+X])$ is *reflexive* if $(x,x) \in \perp$. If all elements are reflexive, we say the frame itself is reflexive. Let $\iota^{\mathsf{r}}\colon \mathsf{IF}^{\mathsf{r}}_{\pm} \rightarrowtail \mathsf{IF}_{\pm}$ be the full subcategory of reflexive frames.

3.2 Free Girard quantales from reflexive implication frames

With the same reasoning as Lemma 2.5, $U^{\perp}_{\pm}$ has cartesian lifts over surjections, therefore by Lemma 2.4 tells us $F^{\otimes}_{\pm}\colon \mathsf{IF}_{\pm} \to \mathsf{PS}_{\pm}$ has a right adjoint, $U^{\otimes}_{\pm}$. This functor behaves just like $U^{\otimes}$ except it operates pointwise on pairs of multisets: given $(X, +, 0, \perp \subseteq X^2) \in \mathsf{PS}_{\pm}$ and $U^{\otimes}_{\pm}(\mathcal{X}) = (X, \perp' \subseteq \mathbb{N}[X+X])$, a pair of multisets $(\Gamma, \Delta) \in \mathbb{N}[X]$ is in $\perp'$ iff the (pairwise) sum $(\sum_i \gamma_i, \sum_j \delta_j)$ is in $\perp$.

Likewise, by the same reasoning as Lemma 2.5, $U^{\perp}_{\dagger}$ has cartesian lifts for counit morphisms in $\mathsf{CMon}^{\dagger}$ of the free involutive commutative monoid $F^{\dagger} \dashv U^{\dagger}$, which are surjections. Thus we can again use Lemma 2.4 to lift the free involutive commutative monoid adjunction $F^{\dagger} \dashv U^{\dagger}$ to the pullback projection map $F^{\neg}\colon \mathsf{PS}_{\pm} \to \mathsf{PS}^{\dagger}$, yielding $U^{\neg}$ which sends an involutive phase space $(X, +, 0, \dagger, \perp \subseteq X)$ to a signed phase space $(X, +, 0, \perp \subseteq X^2)$ with $(x,y) \in \perp'$ whenever $x + y^{\dagger} \in \perp$. The adjunction $F^{\neg} \dashv U^{\neg}$ makes explicit that one consider only left handed sequents as long as one allows element to have a 'dual' element which behaves as if that element were on the other side of the turnstile. The unit, $\eta^{\neg}$, sends the elements of a signed phase space $(X, +, 0, \perp \subseteq X^2)$ into a signed phase space whose elements are thought of as pairs of the original signed phase space $(X^2, +, 0, \perp' \subseteq X^4)$, where the second element of the pair acts as if its being added to the other side of the turnstile, i.e. $((a_+, a_-), (b_+, b_-)) \in \perp'$ iff $(a_+ + b_-,\ b_+ + a_-) \in \perp$. Now, having defined $U^{\otimes}_{\pm}$ and $U^{\neg}$, we now turn to defining $U^{\oplus}_{\pm}\colon \mathsf{GQ} \to \mathsf{PS}^{\dagger}$.

Lemma 3.2. The forgetful functor $U^{\leq}\colon \mathsf{PreCMon} \to \mathsf{CMon}$ is a fibration.

By Lemmas 2.4 and 3.2, we lift the $U^{\dagger} \dashv F_{\dagger}$ adjunction obtain right adjoint to $\pi_{\leq}$. Then, by Lemma 2.6 the composite left adjoint $F^{\vee}_{\dagger}$ induces GQ as a reflective subcategory of a comma category, which (like before) restricts to $\mathsf{PS}^{\dagger}$. So $F^{\oplus}_{\pm}$ is the reflector of this derived subcategory $U^{\oplus}_{\pm}\colon \mathsf{GQ} \rightarrowtail \mathsf{PS}^{\dagger}$.

At this stage, we have a composite left adjoint $F^{\otimes\neg\oplus} := F^{\otimes}_{\pm} \cdot F^{\neg} \cdot F^{\oplus}_{\pm}$ with right adjoint $U^{\otimes\neg\oplus} := U^{\oplus}_{\pm} \cdot U^{\neg} \cdot U^{\otimes}_{\pm}$. We would like to extend this to $\mathsf{IF}^{\mathsf{r}}_{\pm}$; however, although we can map any implication frame to its subframe of reflexive elements, there is no functor $\mathsf{IF}_{\pm} \to \mathsf{IF^{r}}_{\pm}$.[8]

Lemma 3.3. Let $U_{\pm}\colon \mathsf{GQ} \to \mathsf{IF}^{\mathsf{r}}_{\pm}$ send a Girard quantale $\mathcal{Q}$ to the reflexive subframe of $U^{\otimes\neg\oplus}(\mathcal{Q})$ and send a GQ morphism f to the restriction of $U^{\otimes\neg\oplus} f$ to reflexive elements. This is functorial and right adjoint to $F_{\pm} := \iota^{\mathsf{r}} \cdot F^{\otimes\neg\oplus}$.

3.3 Logical interpretation of implication frames

Now that we are working in a signed setting, we can begin to recover concepts introduced in Section 1.

[8] The issue is that the continuity property of morphisms is not preserved when restricting the functions to reflexive elements.

Lemma 3.4. The elements of the Girard quantale $F^{\otimes\neg\oplus}(\mathcal{X})$ are 'roles' $\mathbb{R}$ of Section 1, and the RSR function (Definition 1.3) restricted to roles is precisely the $(-)^{\perp}$ operation of the quantale.

Given $\eta^{\otimes\neg\oplus}_{\mathcal{X}}\colon \mathcal{X} \to \hat{\mathcal{X}}$, we write an arbitrary element of $\hat{X}$ as $\mathtt{A} = \langle \mathtt{a}_+,\ \mathtt{a}_- \rangle$ where $\mathtt{a}_\pm \in \mathcal{P}[\mathbb{N}[X+X]]$ (in particular, $(-)^{\perp\perp}$ closed subsets). We write a multiset of such elements as $\vec{\mathtt{A}} = \sum_{\mathtt{A}\in\hat{X}} \mathtt{A}^{i_\mathtt{A}}$ for $i_\mathtt{A} \in \mathbb{N}$. Because $\hat{\mathcal{X}}$ is a signed incompatibility frame, we have a notion of consequence among multisets of its elements. The $\perp$ relation of $\hat{\mathcal{X}}$, i.e. $\vec{\mathtt{A}} \vDash \vec{\mathtt{B}}$, holds whenever $(\bigotimes_\mathtt{A} \mathtt{a}_+^{i_\mathtt{A}}) \otimes (\bigotimes_\mathtt{B} \mathtt{b}_-^{i_\mathtt{B}}) \subseteq \perp$, which matches the semantic consequence relation of the implication space semantics of Section 1.

$\mathsf{IF}_\pm$ is a subcategory of IF, restricting to sets of the form $X+X$ and morphisms of the form $f+f$. Conservativity in subcategories of IF is determined by interpreting the morphisms in IF. The unit $\eta^{\otimes\neg\oplus}_{\mathcal{X}}$ is conservative (Definition 2.6) by the same argument as Proposition 2.1. Consequently, any restriction of $\eta^{\otimes\neg\oplus}$ (including $\eta_\pm$ or η^{c} of Section 4) is conservative. When we work instead $\eta_\pm$, i.e. restricting to reflexive implication frames, then the elements of $\hat{\mathcal{X}}$ have natural operations (Lemma 3.5) which are closed under the subset of $\mathcal{Q}^2$ restricted to reflexive elements.

Lemma 3.5. Let $\hat{\mathcal{X}} = U_\pm(\mathcal{Q})$ for some Girard quantale $\mathcal{Q}$. These elements (which are pairs of elements of $\mathcal{Q}$) are closed under the quantale operations of the *twisted quantale* $\mathcal{Q}\times\mathcal{Q}^{\mathrm{op}}$, i.e. $\otimes\times\parr$,[9] $\vee\times\wedge$, and $(-)^{\perp}\times(-)^{\perp}$.

We use these operations to define semantic clauses on the elements of $\hat{\mathcal{X}}$ for MALL connectives and show in Fig. 4 that these operations validate the logical rules of linear logic independent of $\mathcal{X}$.[10]

$$[\![A\otimes B]\!] := \langle \mathtt{a}_+\otimes\mathtt{b}_+, \mathtt{a}_-\parr\mathtt{b}_-\rangle \qquad [\![A\oplus B]\!] := \langle \mathtt{a}_+\vee\mathtt{b}_+, \mathtt{a}_-\wedge\mathtt{b}_-\rangle$$

$$[\![A\parr B]\!] := \langle \mathtt{a}_+\parr\mathtt{b}_+, \mathtt{a}_-\otimes\mathtt{b}_-\rangle \qquad [\![A\& B]\!] := \langle \mathtt{a}_+\wedge\mathtt{b}_+, \mathtt{a}_-\vee\mathtt{b}_-\rangle$$

Proposition 3.1. These formulas match the semantic clauses for $\otimes$ and $\oplus$ from Section 1.

Proof. This is immediate, modulo some differences in presentation. For $[\![A\otimes B]\!]^+$: the formula for $\otimes$ matches the $\sqcup$ operation. For $[\![A\otimes B]\!]^-$: the definition of $\parr$ was unfolded in Section 1, and by Lemma 3.4 RSR corresponds to $(-)^{\perp}$. For $[\![A\oplus B]\!]^+$: the join operation of the quantale $F^{\otimes\neg\oplus}(\mathcal{X})$ is union followed by $(-)^{\perp\perp}$ closure, i.e. $\sqcap$. For $[\![A\oplus B]\!]^-$, observe that $x\wedge y = (x^\perp \vee y^\perp)^\perp$ for any elements x, y in a Girard quantale. Again, $\parr$ and & do not need to be checked as in both sets of definitions, the De Morgan laws are satisfied: $[\![A\parr B]\!] := [\![\neg(\neg A\otimes\neg B)]\!]$, and $[\![A\& B]\!] := [\![\neg(\neg A\oplus\neg B)]\!]$. □

For structural rules, we have also deliberately enforced reflexivity. However, our choice of $\perp$ is important for which instances of cut obtain among the atomic sequents. Reflexivity requires $\mathtt{a}_- \subseteq \mathtt{a}_+^\perp$ and, by Fig. 5, cut requires $\mathtt{a}_+^\perp \subseteq \mathtt{a}_-$. This means that the structural rules of classical linear logic demand that $\mathtt{a}_- = \mathtt{a}^{+\perp}$.

Definition 3.6 (Supralinearity and supraclassicality)**.** Given a set of propositional variables X and consequence relation $\vdash$ on the set of MALL formulas on X, we say $\vdash$ is *supralinear* if $\vdash_{\text{MALL}} \subseteq \vdash$, where $\vdash_{\text{MALL}}$ is the MALL provability relation. Analogously, a consequence relation on the set of Boolean formulas on X is *supraclassical* if $\vdash_{\text{classical}} \subseteq \vdash$.

Proposition 3.2. For any $\mathcal{X} = (X,\perp) \in \mathsf{IF}^{\mathsf{r}}_\pm$ (which may fail to satisfy cut) with $\eta_\pm\colon \mathcal{X}\to\hat{\mathcal{X}}$, the consequence relation of $\hat{\mathcal{X}}$ is supralinear, and the atomic sequents that it validates are precisely $\perp$.

[9] Note $x\parr y := (x^\perp\otimes y^\perp)^\perp$, and this operation is a monoidal product for the opposite Girard quantale.

[10] It is interesting that the proofs in Fig. 4 do not rely on the assumption that the elements satisfy reflexivity, despite the operations originating from a search for operations which preserve reflexivity.

$$\boxed{\frac{\Gamma \vdash A,\Delta}{\Gamma,\neg A \vdash \Delta}\neg\mathrm{L}} \quad \begin{aligned} & \pi_2(\langle \mathtt{a}_+,\mathtt{a}_-\rangle) \subseteq (\Gamma_+\Delta_-)^\perp \\ \Longleftrightarrow \quad & \mathtt{a}_- \subseteq (\Gamma_+\Delta_-)^\perp \\ \Longleftrightarrow \quad & \pi_1(\langle \mathtt{a}_-,\mathtt{a}_+\rangle) \subseteq (\Gamma_+\Delta_-)^\perp \end{aligned} \qquad \boxed{\frac{\Gamma,A \vdash \Delta}{\Gamma \vdash \neg A,\Delta}\neg\mathrm{R}} \quad \begin{aligned} & \pi_1(\langle \mathtt{a}_+,\mathtt{a}_-\rangle) \subseteq (\Gamma_+\Delta_-)^\perp \\ \Longleftrightarrow \quad & \mathtt{a}_+ \subseteq (\Gamma_+\Delta_-)^\perp \\ \Longleftrightarrow \quad & \pi_2(\langle \mathtt{a}_-,\mathtt{a}_+\rangle) \subseteq (\Gamma_+\Delta_-)^\perp \end{aligned}$$

$$\boxed{\frac{\Gamma,A,B \vdash \Delta}{\Gamma,A\otimes B \vdash \Delta}\otimes\mathrm{L}} \quad \begin{aligned} & \mathtt{a}_+\mathtt{b}_+ \subseteq \Gamma_+\Delta_-^\perp \\ & \text{(Holds definitionally)} \end{aligned} \qquad \boxed{\frac{\Gamma \vdash A,\Delta \quad \Theta \vdash B,\Omega}{\Gamma,\Theta \vdash A\otimes B,\Delta,\Omega}\otimes\mathrm{R}} \quad \begin{aligned} & (\Gamma_+\Delta_- \subseteq \mathtt{a}_-^\perp)\wedge(\Theta_+\Omega_- \subseteq \mathtt{b}_-^\perp) \\ \Longrightarrow \; & \Gamma_+\Delta_-\Theta_+\Omega_- \subseteq \mathtt{a}_-^\perp\mathtt{b}_-^\perp \\ \Longleftrightarrow \; & \Gamma_+\Delta_-\Theta_+\Omega_- \subseteq (\mathtt{a}_- \parr \mathtt{b}_-)^\perp \end{aligned}$$

$$\boxed{\frac{\Gamma,A \vdash \Delta \quad \Gamma,B \vdash \Delta}{\Gamma,A\oplus B \vdash \Delta}\oplus\mathrm{L}} \quad \begin{aligned} & (\Gamma_+\Delta_- \subseteq a_+^\perp)\wedge(\Gamma_+\Delta_- \subseteq \mathtt{b}_+^\perp) \\ \Longleftrightarrow \; & \Gamma_+\Delta_- \subseteq \mathtt{a}_+^\perp \wedge \mathtt{b}_+^\perp \\ \Longleftrightarrow \; & \Gamma_+\Delta_- \subseteq (\mathtt{a}_+ \vee \mathtt{b}_+)^\perp \end{aligned} \qquad \boxed{\frac{\Gamma \vdash A,\Delta}{\Gamma \vdash A\oplus B,\Delta}\oplus\mathrm{R}} \quad \begin{aligned} & \Gamma_+\Delta_- \subseteq \mathtt{a}_-^\perp \\ \Longrightarrow \; & \Gamma_+\Delta_- \subseteq \mathtt{a}_-^\perp \vee \mathtt{b}_-^\perp \\ \Longleftrightarrow \; & \Gamma_+\Delta_- \subseteq (\mathtt{a}_- \wedge \mathtt{b}_-)^\perp \end{aligned}$$

Figure 4: The logical rules of MALL for $\neg$, $\otimes$, and $\oplus$ are validated for any pairs of elements $\langle \mathtt{a}_+,\mathtt{a}_-\rangle,\langle \mathtt{b}_+,\mathtt{b}_-\rangle$ of $\hat{\mathcal{X}}$ for any implication frame $\mathcal{X}$ with $\eta_{\mathcal{X}}\colon \mathcal{X} \to \hat{\mathcal{X}}$. Note negation is typically written as $(-)^\perp$. The quantale $\otimes$ in the validations on the right are represented via concatenation for space. Invertible steps are denoted with $\Longleftrightarrow$, whereas implications are denoted with $\Longrightarrow$.

$$\frac{\Gamma \vdash A,\Delta \quad \Theta,A \vdash \Omega}{\Gamma,\Theta \vdash \Delta,\Omega}\,\mathrm{Cut}$$

Figure 5: The top expresses $\Gamma_+\Delta_- \subseteq \mathtt{a}_-^\perp$ for some Γ,Δ and that $\mathtt{a}_+ \subseteq (\Theta_+\Omega_-)^\perp$ for some Θ,Ω. The bottom asserts $\Gamma_+\Delta_- \subseteq (\Theta_+\Omega_-)^\perp$. That this entailment holds for all $\Gamma,\Delta,\Theta,\Omega$ is for $\mathtt{a}_-^\perp \subseteq \mathtt{a}_+$, which is true iff $\mathtt{a}_+^\perp \subseteq \mathtt{a}_-$.

Proof. Suppose $\Gamma \vdash_{\mathrm{MALL}} \Delta$. By cut-elimination for MALL [17], $\Gamma \vdash_{\mathrm{MALL}} \Delta$ has a cut-free proof. The base case is that the proof is a single identity rule, which holds in $\hat{\mathcal{X}}$ in virtue of being a reflexive implication frame. Each remaining step in the proof is a logical rule of MALL, which by Fig. 4 holds in $\hat{\mathcal{X}}$. Therefore $\Gamma \vdash_{\mathcal{X}} \Delta$. That the valid atomic sequents are precisely $\perp$ is a restatement that $\eta_\pm$ is conservative. □

4 Idempotent logic

In this section, we restrict the $F_\pm \dashv U_\pm$ adjunction and obtain supraclassical consequence relations (rather than supralinear) parameterized by a choice of implication frame.

Definition 4.1 (Category of containment implication frames)**.** An element x of an implication frame $(X, \perp \subseteq \mathbb{N}[X+X])$ is *idempotent* if $(\{x\},\{\})^\perp = (\{x,x\},\{\})^\perp$ and $(\{\},\{x\})^\perp = (\{\},\{x,x\})^\perp$. Note that, if all elements are idempotent, subsets of $\mathbb{N}[X+X]$ are in bijection with subsets of $\mathcal{P}[X+X]$. Let a *containment implication frame* be an implication where all elements are idempotent and satisfy *containment*, i.e. $(\{x\},\{x\})^\perp = \mathcal{P}[X+X]$. Let $\iota^c\colon \mathsf{IF}^c_\pm \rightarrowtail \mathsf{IF}^r_\pm$ be the subcategory of containment frames.

Demanding containment requires $\perp$ to contain all sequents of the form $\Gamma,x \vdash x,\Delta$.[11] Although we can restrict any reflexive, idempotent implication frame to its subframe of idempotent, containment elements,

[11] In the bilateralist interpretation of the turnstile, these are all of the positions in which some proposition is be both asserted and denied in the same breath. While such 'explicit' contradictions are out of bounds, these frames can still violate monotonicity and cut. If $a \vdash b$ and $b \vdash c$, then $a \wedge \neg c$ (the assertion that $a \nvdash c$) is 'implicitly' contradictory.

this is not a functor $\mathsf{IF}^{\mathsf{r;i}}_{\pm} \to \mathsf{IF}^{\mathsf{c}}_{\pm}$. We will repeat the tactic of Lemma 3.3; however, we need to restrict both the implication frame side and the Girard quantale side of the adjunction.

Definition 4.2 (Category of join-idempotent Girard quantales)**.** Let $\mathsf{GQ}^{\mathsf{ji}}$ be the full subcategory of GQ where we restrict to *join-idempotent Girard quantales*, which are those for which every element $q \in \mathcal{Q}$ can be expressed as some join of idempotent elements of $\mathcal{Q}$.

Lemma 4.1. Let $U^{\mathsf{c}}_{\pm} : \mathsf{GQ}^{\mathsf{ji}} \to \mathsf{IF}^{\mathsf{c}}_{\pm}$ act on join-idempotent Girard quantales just as $U_{\pm}$ does, except with the quantales and their morphisms restricted to idempotent, containment-satisfying elements. This is functorial and right adjoint to $F^{\mathsf{c}}_{\pm} := (\iota^{\mathsf{c}} \cdot F_{\pm})|_{\mathsf{GQ}^{\mathsf{ji}}}$.

Given some $\mathcal{X} := (X, \perp \subseteq \mathcal{P}(X+X))$ satisfying containment, the unit of $F^{\mathsf{c}}_{\pm} \dashv U^{\mathsf{c}}_{\pm}$ is a subframe of $(Q^2, \perp')$, where Q is the underlying set of $\mathcal{Q}$ the free Girard quantale on $\mathcal{X}$. We now investigate sorts of operations make sense on this subset of idempotent and containment-satisfying pairs of quantale elements. First we note that, while the join of two idempotent elements in $\mathcal{Q}$ is not necessarily idempotent, they can be joined in the subquantale of idempotents.

Lemma 4.2. Let $\mathcal{Q} := (Q, \otimes, I, \vee)$ be any quantale. There is an idempotent subquantale $(Q_{\otimes}, \otimes, \tilde{\vee})$ with elements $Q_{\otimes} := \{q \in Q \mid q \otimes q = q\}$, the same $\otimes$ as $\mathcal{Q}$, and (binary) joins are given by $x \;\tilde{\vee}\; y := x \vee y \vee (x \otimes y)$.[12]

The swap operation preserves idempotency and containment satisfaction. Although $\mathcal{Q} \times \mathcal{Q}^{\mathrm{op}}$ provided a natural set of operations on pairs of reflexive quantale elements, the $⅋$ operator cannot be restricted to idempotent elements,[13] so we must consider other natural structures on pairs of quantale elements.

Lemma 4.3. The elements of $\hat{\mathcal{X}} = U^{c}_{\pm}(\mathcal{Q})$ for some Girard quantale $\mathcal{Q}$ are closed under the quantale operations of the *mixed quantale* $\mathcal{Q}_{\otimes} \times \mathcal{Q}^{\vee}_{\otimes}$, where $\mathcal{Q}^{\vee}_{\otimes}$ is the quantale obtained from $\mathcal{Q}_{\otimes}$ with $\tilde{\vee}$ as both its monoidal operation and its join operation.

We can use the mixed quantale multiplication operation to define the semantic formula for $\wedge$, i.e. $[\![A \wedge B]\!] := \langle \mathtt{a}_+ \otimes \mathtt{b}_+,\ \mathtt{a}_- \;\tilde{\vee}\; \mathtt{b}_- \rangle$. We can then use the negation operation to define $[\![\mathtt{A} \vee \mathtt{B}]\!] := \neg(\neg \mathtt{A} \wedge \neg \mathtt{B})$, recovering the semantic clauses of Section 1 for implication space semantics of classical logic.

Proposition 4.1. For any $\mathcal{X} = (X, \perp) \in \mathsf{IF}^{\mathsf{c}}_{\pm}$ with $\eta^{\mathsf{c}} \colon \mathcal{X} \to \hat{\mathcal{X}}$, the consequence relation of $\hat{\mathcal{X}}$ is supraclassical, and the atomic sequents that it validates are precisely $\perp$.

Proof. This result was also presented as a corollary of [1, Thm. 76], which shows that the (contractive) implication space semantics is sound and complete for NMMS, and [1, Prop 25], which shows that, with containment, NMMS is supraclassical. We offer a complementary proof via showing that the Lindenbaum algebra of the semantic consequence relation is a Boolean algebra: any set obeying the Robbins equation $(A \vee B) \wedge (A \vee \neg B) = A$ is a Boolean algebra [21].[14] $[\![(A \vee B) \wedge (A \vee \neg B)]\!]$ is a pair of quantale elements, $\langle a_+ \otimes (I \vee b_+ \vee b_-), \mathtt{a}_- \otimes (\mathtt{b}_- \vee \mathtt{b}_+) \rangle$. The semantic value of $[\![(A \vee B) \wedge (A \vee \neg B)]\!]$ is analyzed below, with the first element on the left and second element on the right:[15]

[12] This is a special case of general joins: $\tilde{\bigvee}_{i \in I} x_i := \bigvee_{\varnothing \subset J \subseteq_{\mathrm{fin}} I} \bigotimes_{j \in J} x_j$.

[13] Note that idempotent elements are not closed under $(-)^{\perp}$. Also, $\perp$ itself might not be an idempotent element, hence $Q_{\otimes}$ is not naturally a Girard quantale.

[14] We have rewritten the original equation: $\neg(A \vee B) \vee \neg(A \vee \neg B) = \neg A$ using De Morgan duality and double negation elimination. Also, $\vee$ must be commutative and associative, which holds in our case because it consists in $\otimes$ in one component and $\tilde{\vee}$ in the other component, both of which are commutative and associative.

[15] For space we represent $\otimes$ by concatenation. We also liberally apply idempotence of $\otimes$.

$(\mathtt{a}_+ \vee \mathtt{b}_+ \vee (\mathtt{a}_+\mathtt{b}_+))(\mathtt{a}_+ \vee \mathtt{b}_- \vee (\mathtt{a}_+\mathtt{b}_-))$		$(\mathtt{a}_-\mathtt{b}_-) \vee (\mathtt{a}_-\mathtt{b}_+) \vee (\mathtt{a}_-\mathtt{b}_+\mathtt{b}_-)$	
$\mathtt{a}_+ \vee \mathtt{a}_+\mathtt{b}_+ \vee \mathtt{a}_+\mathtt{b}_- \vee \mathtt{a}_+\mathtt{b}_+\mathtt{b}_- \vee \mathtt{b}_+\mathtt{b}_-$	Distributivity	$\mathtt{a}_-(\mathtt{b}_- \vee \mathtt{b}_+ \vee \mathtt{b}_+\mathtt{b}_-)$	Factor out $\mathtt{a}_-$
$\mathtt{a}_+(I \vee \mathtt{b}_+ \vee \mathtt{b}_- \vee \mathtt{b}_+\mathtt{b}_-) \vee \mathtt{b}_+\mathtt{b}_-$	Factor out $\mathtt{a}_+$	$\mathtt{a}_-(\mathtt{b}_- \vee \mathtt{b}_+)$	Containment
$\mathtt{a}_+(I \vee \mathtt{b}_+ \vee \mathtt{b}_-)$	Containment		

Now if we try to prove $(A \vee B) \wedge (A \vee \neg B) \vDash A$, we check if $(a_+ \otimes (I \vee \mathtt{b}_+ \vee \mathtt{b}_-)) \otimes a_- \leq \bot$, which is true because of containment: $a_+ \otimes a_- \otimes (I \vee \mathtt{b}_+ \vee \mathtt{b}_-) = 0$. Likewise, $A \vDash (A \vee B) \wedge (A \vee \neg B)$ amounts to checking $a_+ \otimes (a_- \otimes \mathtt{b}_- \vee \mathtt{b}_+) \leq \bot$, which also holds due to containment. Therefore, independent of our choice of $\bot$ in our starting containment implication frame, we satisfy the structural laws of classical (multisuccedent) logic, and we also satisfy the logical laws too in virtue of being a Boolean algebra. □

5 Conclusion

We have shown the implication space semantics actually arises naturally as unit to an adjunction between reflexive incompatibility frames and Girard quantales (resp: containment incompatibility frames and join-idempotent Girard quantales). The semantic clauses for linear logic formulas arise naturally as operations on pairs of quantale elements that preserve reflexivity, while the classical logic formulas likewise arise on pairs of idempotent quantale elements that preserve containment-satisfaction.

In [1], the non-contractive and contractive implication space semantics are told as related but disjoint stories; however, we have demonstrated that the contractive story derives from an adjunction which is a restriction of the adjunction characterizing the non-contractive story. The difference between non-contractive variants of the $\wedge R$ and $\vee L$ NMMS rules and their contractive counterparts (which include a third top sequent) is explained by them sharing the form of a quantale join, where the extra top sequent comes from the form of joins in an idempotent subquantale. We also showed that, by starting with a consequence relation which validates contraction and containment, the same logical elaboration process conservatively expands a vocabulary with classical logic formulas and produces a supraclassical logic.

Software implementation: ROLE.jl [22] is a Julia library which implements the concepts in this paper, in particular the declaration of (idempotent) implication frames $\mathcal{X} := (X, \bot)$ and the evaluation of semantic consequence between elements of $\hat{\mathcal{X}}$, specified by logically complex formulas. The tests in this repository are also a source of worked examples of implication frames and the implication space semantics which are too large to present in this current work.

Future work: A formal statement relating between traditional categorical logic and logical expressivism would account for their various (dis)similarities. NMMS and implication space semantics are intended to generalize various forms of *propositional* logic; however, there are many directions of possible generalization. Informally, we could say implications frames are enriched in boolean values, but enriching in sets (yielding proof-relevant predicate logics) or real numbers (probabilistic frames) would be particularly valuable if they are to model social practices that regulate allowable inferences. Framing the underlying data structures and operations categorically is a key first step to this. Furthermore, more exploration is needed of the possible applications of (co)limits and other structures within the category of implication frames (or the Kleisli category of the adjunctions we've constructed), as it is important for the computational scalability of ROLE.jl that frames which consist of largely disjoint vocabularies can be implicitly represented as colimits.

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A Lemmas

Lemma 2.1

Proof. Following [23]: $(-)^\perp$ is a antitone Galois connection on $\mathcal{P}[X]$. Therefore the fix points of $(-)^\perp$ are precisely the subsets of X which are equal to $(-)^\perp$ applied to some other subset. $\perp = \{0\}^\perp$, hence $\perp = \perp^{\perp\perp}$. □

Lemma 2.2

Proof. We'll construct a bijection. Let f send $\mathcal{X} := (X, +, 0, \perp \subseteq X)$ (with implicit $\leq_\mathcal{X}$ order) to $(\mathcal{X}, \mathrm{Gir}(\mathcal{X}), q_\mathcal{X})$. As shown in Definition 2.2, $q_\mathcal{X}$ is both surjective and an order embedding when restricted to the principal lower sets, which is what $\tilde{q}_\mathcal{X}$ is.[16] Then f^{-1} takes the data $(\mathcal{P}, \mathcal{Q}, \phi\colon F^\vee(\mathcal{P}) \twoheadrightarrow \mathcal{Q})$ and returns $\mathcal{P}$ equipped with the lower set $\phi_*(\perp) \in F^\vee(\mathcal{P})$, which is a lower set computed by applying the right adjoint of ϕ to $\perp$. This means $\phi_*(\perp) = \bigcup(\{A \in F^\vee(\mathcal{P}) \mid \phi(A) \leq \perp\})$, i.e. $p \in \perp' \iff \phi(p^\downarrow) \subseteq \perp$.

First we must show that $f \cdot f^{-1} = \mathrm{id}$. Start with a phase space $\mathcal{X} := (X, +, 0, \perp)$. We return back to $\mathcal{X}$ iff $q_{\mathcal{X},*}(\perp) = \perp$. Because $x \in \perp \iff x^{\perp\perp} \subseteq \perp \iff q_\mathcal{X}(x^{\perp\perp}) \subseteq \perp \iff x \in \perp'$, we have $\perp = \perp'$.

Last we must show that $f^{-1} \cdot f = \mathrm{id}$. Start with a monoidal preorder $\mathcal{P} := (P, +, 0, \leq_\mathcal{P})$, a Girard quantale $\mathcal{Q} := (Q, \otimes, \leq_\mathcal{Q}, \vee, \perp)$, and a quantale morphism $\phi\colon F^\vee(\mathcal{P}) \twoheadrightarrow \mathcal{Q}$ such that $\tilde{\phi}$ is an order embedding. We must recover all this data just from applying f to $\mathcal{X} := (P, +, 0, \perp' := \phi_*(\perp))$. We first want to know how to compute when elements are in $\perp'$:

$$
\begin{array}{rr}
a + x \in \perp' & \text{Defn of } (-)^\perp \text{ in phase space} \\
\phi((a+x)^\downarrow) \leq_\mathcal{Q} \perp & \text{Defn of } \perp' \\
\phi(a^\downarrow \otimes_{F^\vee\mathcal{P}} x^\downarrow) \leq_\mathcal{Q} \perp & \text{Day is monoidal} \\
\phi(x^\downarrow) \otimes_\mathcal{Q} \phi(a^\downarrow) \leq_\mathcal{Q} \perp & \phi \text{ is monoidal} \\
\tilde{\phi}(x) \otimes_\mathcal{Q} \tilde{\phi}(a) \leq_\mathcal{Q} \perp & \text{Defn of } \tilde{\phi}
\end{array}
$$

Call this result **L1**. Now we recover the order, $\leq_\mathcal{P}$.

$$
\begin{array}{rl}
a \leq_\mathcal{P} b & \\
\tilde{\phi}(a) \leq_\mathcal{Q} \tilde{\phi}(b) & \tilde{\phi} \text{ is an embedding} \\
\tilde{\phi}(a) \otimes_\mathcal{Q} \bigvee(\{c \in Q \mid c \otimes \tilde{\phi}(b) \leq_\mathcal{Q} \perp\}) \leq_\mathcal{Q} \perp & \text{Defn of } (-)^\perp \\
\bigvee(\{\tilde{\phi}(a) \otimes_\mathcal{Q} c \mid c \otimes_\mathcal{Q} \tilde{\phi}(b) \leq_\mathcal{Q} \perp\}) \leq_\mathcal{Q} \perp & \text{Distributivity of } \otimes,\ \vee \\
\forall c \in Q\colon c \otimes_\mathcal{Q} \tilde{\phi}(b) \leq_\mathcal{Q} \perp \implies \tilde{\phi}(a) \otimes_\mathcal{Q} c \leq_\mathcal{Q} \perp & \text{Property of } \vee \\
\forall c \in P\colon \tilde{\phi}(c) \otimes_\mathcal{Q} \tilde{\phi}(b) \leq_\mathcal{Q} \perp \implies \tilde{\phi}(c) \otimes_\mathcal{Q} \tilde{\phi}(a) \leq_\mathcal{Q} \perp & c = \bigvee_i \tilde{\phi}(c_i) \text{ for } c_i \in P \text{ b/c } \phi \text{ surj.} \\
\forall c \in P\colon c + b \in \perp' \implies c + a \in \perp' & \textbf{L1} \\
a \leq_\mathcal{X} b & \text{Defn of } \leq_\mathcal{X}
\end{array}
$$

[16] Recall that for any quantale morphism $\phi\colon F^\vee(\mathcal{P}) \to \mathcal{Q}$, by the hom set bijection of $F^\vee \dashv U^\vee$ we have a corresponding symmetric monoidal preorder morphism $\tilde{\phi}\colon \mathcal{P} \to U^\vee(\mathcal{Q})$.

We need to show that $\mathrm{Gir}(\mathcal{X}) \cong \mathcal{Q}$. We now show $x \in A^{\perp}$ (in $\mathcal{X}$) iff $x \in \phi_*(\phi(A)^{\perp})$.

$\forall a\colon\ a+x \in \perp'$	Defn of $(-)^{\perp}$ in phase space
$\forall a\colon\ \phi(x^{\downarrow}) \otimes \phi(a^{\downarrow}) \leq_Q \perp$	**L1**
$\forall a\colon\ \phi(x^{\downarrow}) \leq_Q \phi(a)^{\perp}$	$\perp$ is dualizing
$\phi(x^{\downarrow}) \leq_Q \phi(A)^{\perp}$	$\phi(A)^{\perp}$ as a meet
$x^{\downarrow} \leq_{F^{\vee}\mathcal{P}} \phi_*(\phi(A)^{\perp})$	$\phi \dashv \phi_*\colon\ \phi(a) \leq_Q b \iff a \leq_{F^{\vee}\mathcal{P}} \phi_*(b)$
$x^{\downarrow} \subseteq \phi_*(\phi(A)^{\perp})$	Defn of $\leq_{F^{\vee}\mathcal{P}}$
$x \in \phi_*(\phi(A)^{\perp})$	Lower set embedding

Call this **L2**. Now we can compute $A^{\perp\perp}$ after noting that the image of ϕ for a Galois connection is the set of fix points of $\phi_* \cdot \phi$, and the surjectivity of ϕ means that for every element $q \in Q$ we have $\phi(\phi_*(q)) = q$.

$A^{\perp\perp}$	
$\phi_*(\phi(\phi_*(\phi(A)^{\perp}))^{\perp})$	**L2**
$\phi_*(\phi(A)^{\perp\perp})$	$\phi_* \cdot \phi = \mathrm{id}_Q$
$\phi_*(\phi(A))$	$(-)^{\perp}$ is dualizing

Letting $\mathrm{Gir}(\mathcal{X})$ be the set of $(-)^{\perp\perp}$ closed lower sets of $\mathcal{X}$, we have established its elements are the fixed points of $\phi \cdot \phi_*$, which is the closure operator associated with the Galois connection. Its fixed points are precisely the elements in the image of ϕ_*. Therefore restricting the domain of ϕ and the codomain of ϕ_* makes a join and $\otimes$-preserving bijection between $\mathrm{Gir}(\mathcal{X})$ and $\mathcal{Q}$.

□

Lemma 2.3

Proof. We actually show the following four conditions are equivalent:

1. $\forall A \subseteq X\colon\ f(A)^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}} = f(A^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}})^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}}$
2. $\forall A \subseteq X\colon\ f(A^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}) \subseteq f(A)^{\perp_Y\perp_Y}$
3. $\forall y \in Y\colon\ f^{-1}(y^{\perp_{\mathcal{Y}}})^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}} \subseteq f^{-1}(y^{\perp_{\mathcal{Y}}})$
4. $\forall A, B \subseteq X\colon\ A^{\perp_{\mathcal{X}}} \subseteq B^{\perp_{\mathcal{X}}} \implies f(A)^{\perp_{\mathcal{Y}}} \subseteq f(B)^{\perp_{\mathcal{Y}}}$

1 $\implies$ **2**: Because $(-)^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}$ is increasing, we automatically have $f(A)^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}} \subseteq f(A^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}})^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}}$, and applying **1** to the right hand side yields **2**.

2 $\implies$ **1**: Apply $(-)^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}}$ to both sides. Because it is idempotent, that yields that $f(A)^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}}$ is a superset of $f(A^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}})^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}}$. That it is a subset follows from $(-)^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}$ being increasing.

2 $\implies$ **3**: Let $A := f^{-1}(y^{\perp_{\mathcal{Y}}})$. This means $f(A) \subseteq y^{\perp_{\mathcal{Y}}}$. Applying $(-)^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}}$ to both sides (and noting the RHS is already closed) we get that **(2a.)** $f(A)^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}} \subseteq y^{\perp_{\mathcal{Y}}}$. Let $a' \in A^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}$: we need to show $a' \in A$ in order to prove that preimages of facts are facts. Then $f(a') \in f(A^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}})$, therefore continuity tells us **(2b.)** $f(a') \in f(A)^{\perp_{\mathcal{Y}}\perp_{\mathcal{Y}}}$. Composing **(2b)** and **(2a)** gives us $f(a') \in y^{\perp_{\mathcal{Y}}}$, and applying f^{-1} shows that $a' \in f^{-1}(y^{\perp_{\mathcal{Y}}}) = A$.

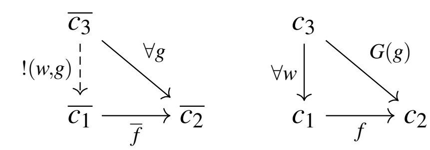
3 $\implies$ **4**: We pick an arbitrary $y \in f(A)^{\perp}$ and show that $y \in f(B)^{\perp}$. First, we have **(3a.)** $y \in f(A)^{\perp} \iff \forall a\colon f(a)+y \in \perp_{\mathcal{Y}} \iff A \subseteq f^{-1}(y^{\perp_{\mathcal{Y}}})$. From the assumption $A^{\perp} \subseteq B^{\perp}$, we apply $(-)^{\perp_{\mathcal{Y}}}$ to both sides to obtain $B \subseteq B^{\perp\perp} \subseteq A^{\perp\perp}$. Therefore B is contained in every closed set containing A. By **3**, $f^{-1}(y^{\perp_{\mathcal{Y}}})$ is a closed set, so $B \subseteq f^{-1}(y^{\perp_{\mathcal{Y}}})$ and by **3a** $y \in f(B)^{\perp}$.

4 $\implies$ **2**: by letting $B = A^{\perp\perp}$ (the antecedent becomes true because $x^{\perp\perp\perp} = x^{\perp}$, and the consequent becomes $f(A)^{\perp} \subseteq f(A^{\perp\perp})^{\perp}$, which is tantamount to the statement we are trying to prove because $(-)^{\perp}$ is order reversing).

□

Lemma 2.4

Notation: Given a functor $G\colon \mathsf{B} \to \mathsf{C}$, the cartesian lift of a morphism $f\colon c_1 \to c_2$ is choice of a morphism $\overline{f} \in G^{-1}(f)$, with $\overline{f}\colon \overline{c_1} \to \overline{c_2}$. This satisfies the property that, for every $g\colon \overline{c_3} \to \overline{c_2}$ in B and $w\colon c_3 \to c_1$ such that $w \cdot f = G(g)$, there exists a unique $\overline{w} \in G^{-1}(w)$ such that $\overline{w} \cdot \overline{f} = g$. We refer to this unique morphism induced by g and w as $!(g,w)$.



Proof. To show that $R(b) = (UG(b), \mathrm{dom}(\overline{\varepsilon_{G(b)}}))$ is an object in $\mathsf{A} \times_{\mathsf{C}} \mathsf{B}$ we need that F applied to $UG(b)$ is equal to G applied to $\mathrm{dom}(\overline{\varepsilon_{G(b)}})$. This follows because $G(\mathrm{dom}(\overline{\varepsilon_{G(b)}})) = \mathrm{dom}(\varepsilon_{G(b)}) = FUG(b)$. For morphisms, R sends $f\colon b_1 \to b_2$ to a morphism $(UG(b_1), \mathrm{dom}(\overline{\varepsilon_{G(b_1)}})) \to (UG(b_2), \mathrm{dom}(\overline{\varepsilon_{G(b_2)}}))$ given by $(UG(f), !(\overline{\varepsilon_{G(b_1)}} \cdot f, FUG(f)))$.

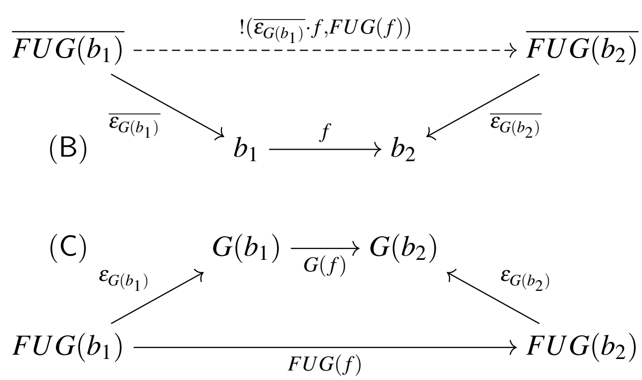


Figure 6: The B component of $R(f)$ for a morphism $f \in \mathsf{B}$ via the universal property of the cartesian lift of $G(f)$. These morphisms, drawn with a dashed arrow, are uniquely characterized by having the property of being postcomposed with $\overline{\varepsilon_{G(\mathrm{cod}(f))}}$ to yield a morphism equal to $\overline{\varepsilon_{G(\mathrm{dom}(f))}} \cdot f$ *and* being equal to $FUG(f)$ in C after applying G. Note the bottom square commutes (necessary to apply the universal property of !) by naturality of $\varepsilon\colon FU \to 1_{\mathsf{C}}$.

Now we've defined a functor R. To show it is right adjoint to π_B, we need to define the unit and counit transformations. The counit is a natural transformation $\varepsilon'\colon \pi_B R \to 1_{\mathsf{B}}$. We need a family of morphisms $\varepsilon'_b\colon \mathrm{dom}(\overline{\varepsilon_{G(b)}}) \to b$. These are the morphisms $\overline{\varepsilon_{G(b)}}$, which are natural with the relevant commutative square being the upper square in Fig. 6. The unit is a natural transformation $\eta' \colon 1_{\mathsf{A}\times_{\mathsf{C}}\mathsf{B}} \to R\pi_B$ where $\eta'_{(a,b)} \colon (a,b) \to (UG(b), \mathrm{dom}(\overline{\varepsilon_{G(b)}}))$. The A component of $\eta'_{(a,b)}$ is $\eta_a \colon a \to UF(a)$, which is equal to $a \to UG(b)$. The B component of $\eta'_{(a,b)}$ is $!(\mathrm{id}_{\mathsf{b}}, F(\eta_a))$.

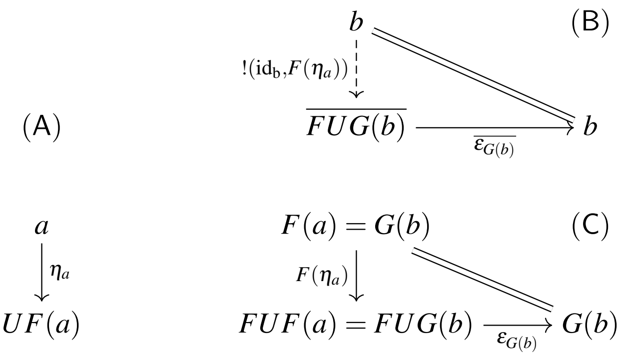


Figure 7: The B component of η'. These are morphisms characterized by having the property of composing with $\overline{\varepsilon_{G(b)}}$ to yield id_b and living over $F(\eta_a)$.

To show η' is natural, let (f,g) be an arbitrary $\mathsf{A}\times_{\mathsf{C}}\mathsf{B}$ morphism $(a,b) \to (a',b')$. Then $R\pi_B(f,g)$ is a morphism $(UG(b), \mathrm{dom}(\overline{\varepsilon_{G(b)}})) \to (UG(b'), \mathrm{dom}(\overline{\varepsilon_{G(b')}}))$ and is equal to $(UG(g), !(\overline{\varepsilon_{G(b)}} \cdot f, FUG(g)))$. We show that the two possible compositions are equal by considering the A component and B components separately.

For the A component, we get $f \cdot \eta_{a'} = \eta_a \cdot UG(g)$ which are equal due to naturality of $\eta \colon 1_{\mathsf{A}} \to UF$ and noting $G(g) = F(f)$. For the second component, we have $g \cdot !(\mathrm{id}_{\mathsf{b}'}, F(\eta_{a'})) \stackrel{?}{=} !(\mathrm{id}_{\mathsf{b}}, F(\eta_a)) \cdot !(\overline{\varepsilon_{G(b)}} \cdot f, FUG(g))$. One way to show these two B morphisms are equal are that they live over the same morphism in C and that they are equal after composing with a cartesian morphism. First we show that they are equal as morphisms in C. Applying G to both sides yields $G(g) \cdot F(\eta'_a)$ (rewritten to $F(f \cdot \eta'_a)$) and $F(\eta_a \cdot UG(g)) = F(\eta_a \cdot UF(f))$ - these are also equal due to naturality of η.

The last step of showing η' is natural is showing the two B morphisms are the same after postcomposing with a cartesian morphism (say: $\overline{\varepsilon_{b'}}$). As they were defined in the figure above, postcomposition with the B component of with this morphism is equal to id'_b. So $g \cdot !(\mathrm{id}_{\mathsf{b}'}, F(\eta_{a'})) \cdot \overline{\varepsilon_{b'}} = g \cdot \mathrm{id}'_b = g$ and

$$\begin{aligned}
&!(\mathrm{id}_{\mathsf{b}}, F(\eta_a)) \cdot !(\overline{\varepsilon_{G(b)}} \cdot f, FUG(g)) \cdot \overline{\varepsilon_{b'}} && \\
&!(\mathrm{id}_{\mathsf{b}}, F(\eta_a)) \cdot \overline{\varepsilon_{G(b)}} \cdot g && \text{See Fig. 6.} \\
&id_b \cdot g && \text{See Fig. 7.} \\
&g &&
\end{aligned}$$

Lastly, after having established natural transformations η' and ε', we verify the triangle equations. First we have $\pi_B(\eta'_{(a,b)}) \cdot \varepsilon'_{\pi_B(a,b)}$ to be id_b. This is $!(\mathrm{id}_b, F(\eta_a)) \cdot \overline{\varepsilon_{G(b)}}$ which is defined in the top commutative triangle in the figure above to compose to the identity id_b.

The second triangle identity requires showing $\eta'_{R(b)} \cdot R(\varepsilon'_b)$ is $\mathrm{id}_{R(b)}$. We'll treat the A and B components independently. $\pi_A(\eta'_{R(b)}) = \eta_{\pi_A(R(b))} = \eta_{UG(b)}$. And $\pi_A(R(\varepsilon'_b)) = UG(\varepsilon'_b) = UG(\overline{\varepsilon_{G(b)}}) = U(\varepsilon_{G(b)})$. That this composite A component, $\eta_{U(x)} \cdot U(\varepsilon_x)$ (letting $x = G(b)$), is the identity follows from one of the triangle identities for $F \dashv U$. For the B components, $\pi_B(\eta'_{R(b)} \cdot R(\varepsilon'_b))$, we first consider what applying U yields. First we have $U(\pi_B(\eta'_{R(b)})) = U(!(\ldots, F(\eta_{\pi_A(R(b))}))) = F(\eta_{\pi_A(R(b))}) = F(\eta_{UG(b)})$, and we also have $U(\pi_B(R(\varepsilon'_b))) = FUG(\varepsilon'_b) = FUG(\overline{\varepsilon_{G(b)}}) = FU(\varepsilon_{G(b)})$. Composing these, we get $F(\eta_{UG(b)}) \cdot FU(\varepsilon_{F(a)})$ and, letting $x = G(b) = F(a)$ we have $F(\eta_{U(x)} \cdot U(\eta_x))$ which is an identity morphism by a triangle identity of $F \dashv U$. So applying U yields the same result as applying U to $\mathrm{id}_{\mathrm{dom}(\overline{\varepsilon_{G(b)}})}$, namely $\mathrm{id}_{FUG(b)}$. We lastly need to check (in order to verify that this composite B component is equal to $\mathrm{id}_{\mathrm{dom}(\overline{\varepsilon_{G(b)}})}$) that it has the same result when postcomposed with a cartesian morphism (say: $\overline{\varepsilon_{G(b)}}$).

$$
\begin{array}{rr}
\pi_B(\eta'_{R(b)} \cdot R(\varepsilon'_b)) \cdot \overline{\varepsilon_{G(b)}} & \\
\pi_B(\eta'_{R(b)}) \cdot \pi_B(R(\varepsilon'_b)) \cdot \overline{\varepsilon_{\mathrm{cod}(\varepsilon'_b)}} & \mathrm{cod}(\varepsilon'_b) = b \\
\pi_B(\eta'_{R(b)}) \cdot \overline{\varepsilon_{G(\mathrm{dom}(\varepsilon'_b))}} \cdot \varepsilon'_b & \text{Property of } R(f) \text{ (Fig. 6)} \\
\pi_B(\eta'_{R(b)}) \cdot \overline{\varepsilon_{G\pi_B R(b)}} \cdot \varepsilon'_b & \pi_B R(b) = \mathrm{dom}(\overline{\varepsilon_{G(b)}}) = \mathrm{dom}(\varepsilon'_b) \\
\mathrm{id}_{\mathrm{dom}(\overline{\varepsilon_{G(b)}})} \cdot \varepsilon'_b & \text{Property of } \eta' \text{ (Fig. 7)} \\
\overline{\varepsilon_{G(b)}} & \text{Defn of } \varepsilon'
\end{array}
$$

□

Lemma 2.5

Proof. We show the forgetful functor $U^{\perp}_{+}\colon \mathsf{PS} \to \mathsf{CMon}$ has cartesian lifts for surjections in CMon. Given any phase space $\mathcal{Y} := (Y, +_Y, 0_Y, \perp_Y)$, monoid $\mathcal{X} := (X, +_X, 0_X)$ and surjective monoid homomorphism $f\colon \mathcal{X} \twoheadrightarrow U^{\perp}(\mathcal{Y})$, there is a cartesian lift $\hat{f}\colon \hat{\mathcal{X}} \to \mathcal{Y}$ with $\hat{\mathcal{X}} := (X, +_X, 0_X, f^{-1}(\perp_Y))$ (because a cartesian lift must also satisfy $U^{\perp}(\hat{f}) = f$ and $U^{\perp}$ is faithful, we can identify $\hat{f}$ with f). First we need to verify this is a morphism in $\mathsf{PS}_{\perp}$: it is already a monoid homomorphism, and we need to check $\perp$ is reserved, which follows from $f(f^{-1}(\perp_Y)) \subseteq \perp_Y$. We also need to check that it is continuous:

$$
\begin{array}{cl}
A^{\perp_{\mathcal{X}}} \subseteq B^{\perp_{\mathcal{X}}} & \\
f^{-1}(f(A)^{\perp_{\mathcal{Y}}}) \subseteq f^{-1}(f(B)^{\perp_{\mathcal{Y}}}) & A^{\perp_{\mathcal{X}}} = \{x \mid \forall a \in A\colon f(a) + f(x) \in \perp_{\mathcal{Y}}\} = f^{-1}(f(A)^{\perp_{\mathcal{Y}}}) \\
f(A)^{\perp_{\mathcal{Y}}} \subseteq f(B)^{\perp_{\mathcal{Y}}} & \text{For surjective } f,\ f^{-1}(A) \subseteq f^{-1}(B) \implies A \subseteq B
\end{array}
$$

To show $f\colon \hat{\mathcal{X}} \to \mathcal{Y}$ is cartesian, let $\mathcal{Z} := (Z, +_Z, 0_Z, \perp_Z)$ be a phase space with a morphism $g\colon \mathcal{Z} \to \mathcal{Y}$ and $w\colon U^{\perp}(\mathcal{Z}) \to \mathcal{X}$ be a monoid morphism such that $w \cdot f = g$.[17] Cartesianness of f is that w is a PS morphism $\mathcal{Z} \to \mathcal{X}$. We must show that w preserves $\perp$. We have $f(w(\perp_Z)) \subseteq \perp_Y$ because g is a PS morphism. Then we can apply f^{-1} to both sides to obtain $w(\perp_Z) \subseteq f^{-1}(\perp_Y) = \perp_X$.

We lastly show w is continuous. We can express $w(A)^{\perp_{\mathcal{X}}}$ as $\{x \mid \forall a \in A\colon g(a) + f(x) \in \perp_{\mathcal{Y}}\} = f^{-1}(g(A)^{\perp_{\mathcal{Y}}})$. We show continuity for arbitrary $A, B \subseteq Z$ with $A^{\perp_{\mathcal{Z}}} \subseteq B^{\perp_{\mathcal{Z}}}$. Note because g is continuous we have $g(A)^{\perp_{\mathcal{Y}}} \subseteq g(B)^{\perp_{\mathcal{Y}}}$. We can then apply f^{-1} which preserves inclusions to get $f^{-1}(g(A)^{\perp_{\mathcal{Y}}}) \subseteq f^{-1}(g(B)^{\perp_{\mathcal{Y}}})$ which is precisely the desired $w(A)^{\perp_{\mathcal{X}}} \subseteq w(B)^{\perp_{\mathcal{X}}}$. □

[17] Again, we use faithfulness to identify the PS morphism g with its underlying monoid morphism. Faithfulness of $U^{\perp}$ is also why we only need to show existence of w, not uniqueness.

Lemma 2.6

Proof. R sends $b \mapsto (UG(b), b, \varepsilon_{G(b)})$ and f to (UGf, f). This is an adjoint with identities for counit morphisms ε' and unit morphisms $\eta'(a,b,f) \mapsto (\eta_a \cdot Uf, \mathrm{id}_b)$. We must for show these are morphisms $(a,b,f) \mapsto R(b)$ in $F \downarrow G$, i.e. the square below commutes:

$$\begin{array}{ccc} F(a) & \xrightarrow{F(\eta_a \cdot Uf)} & FUG(b) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \varepsilon_{G(b)}} \\ G(b) & = & G(b) \end{array}$$

$$\begin{array}{rl} F(\eta_a) \cdot FUf \cdot \varepsilon_{G(b)} & \\ F\eta_a \cdot \varepsilon_{F(a)} \cdot f & \text{Naturality of } \varepsilon \text{ applied to } f\colon F(a) \to U(b) \\ f & \text{Triangle identity} \end{array}$$

We also need to show these morphisms form a natural transformation. Let $(\phi, \psi)\colon (a,b,f) \to (a',b',f')$ be an arbitrary morphism in $F \downarrow G$. We need to verify $(\phi, \psi) \cdot \eta_{(a',b',f')} = \eta_{(a,b,f)} \cdot R(\psi)$. Unpacking these definitions, we have: $(\phi \cdot (\eta_{a'} \cdot Uf'), \psi \cdot id_{b'}) = ((\eta_a \cdot Uf) \cdot (UG\psi), \mathrm{id}_b \cdot \psi)$. Clearly the second component is equal, so we focus on the first:

$$\begin{array}{rl} \phi \cdot \eta_{a'} \cdot Uf' & \\ \eta_a \cdot UF\phi \cdot Uf' & \text{Naturality of } \eta \\ \eta_a \cdot U(F\phi \cdot f') & \text{Functoriality of } U \end{array}$$

$$\begin{array}{rl} (\eta_a \cdot Uf) \cdot (UG\psi) & \\ \eta_a \cdot U(f \cdot G\psi) & \text{Functoriality of } U \\ \eta_a \cdot U(F\phi \cdot f') & (\phi, \psi) \text{ is a } F \downarrow G \text{ morphism} \end{array}$$

Now we show a triangle identity:

$$\begin{array}{rl} \pi_{\mathsf{B}}(\eta'_{(a,b,f)}) \cdot \varepsilon'_{\pi_{\mathsf{B}}(a,b,f)} & \\ \pi_{\mathsf{B}}(\eta'_{(a,b,f)}) \cdot \varepsilon'_b & \text{Defn of } \pi_{\mathsf{B}} \\ \pi_{\mathsf{B}}((\eta_a \cdot Uf, \mathrm{id}_b)) \cdot \mathrm{id}_b & \text{Defn of } \eta' \\ \mathrm{id}_b \cdot \mathrm{id}_b & \text{Defn of } \pi_{\mathsf{B}} \end{array}$$

And the second triangle identity:

$$\begin{array}{rl} \eta'_{R(b)} \cdot R(\varepsilon'_b) & \\ \eta'_{(UG(b), b, \varepsilon_{G(b)})} \cdot R(\mathrm{id}_b) & \text{Defns of } \varepsilon',\ R \\ (\eta_{UG(b)} \cdot U\varepsilon_{G(b)},\ \mathrm{id}_b) \cdot R(\mathrm{id}_b) & \text{Defn of } \eta' \\ (\eta_{UG(b)} \cdot U\varepsilon_{G(b)},\ \mathrm{id}_b) \cdot (\mathrm{id}_{UG(b)}, \mathrm{id}_b) & \text{Defn of } R \\ (\mathrm{id}_{UG(b)},\ \mathrm{id}_b) \cdot (\mathrm{id}_{UG(b)}, \mathrm{id}_b) & \text{Triangle identity} \end{array}$$

Therefore, $\pi_B \dashv R$. Because the counit morphisms are isomorphisms, $B \rightarrowtail (F \downarrow G)$ is a reflective subcategory. □

Lemma 2.7

Proof. By Lemma 2.6, $R\colon \mathsf{GQ} \rightarrowtail (F^{\vee} \downarrow U^{\perp})$, where $F^{\vee} \dashv U^{\vee}$ is the free join completion on a preordered monoid and $U^{\perp}$ is the forgetful functor $\mathsf{GQ} \to \mathsf{Quant}$ discarding dualizing information from a quantale. We need to show that R restricts to the full subcategory $\iota\colon \mathsf{PS} \rightarrowtail (F^{\vee} \downarrow U^{\perp})$. This is a matter of checking that for any Girard quantale $\mathcal{Q}$, we have $R(\mathcal{Q})$ equal to $\iota(\mathcal{X})$ for some phase space $\mathcal{X}$. The general formula for R is $(U^{\vee}U^{\perp}(\mathcal{Q}), \mathcal{Q}, \varepsilon_{U^{\perp}(\mathcal{Q})})$. Because ε is surjective, Lemma 2.2 shows how we can view this triple as just a particular preordered monoid $U^{\vee}U^{\perp}(\mathcal{Q})$ equipped with the lower set $\varepsilon_*(\perp)$. We must now show that $a \leq_{\mathcal{Q}} b$ iff $\forall c \in Q\colon c + b \in \varepsilon_*(\perp) \implies c + a \in \varepsilon_*(\perp)$.

In the forward direction, by monotonicity of $\leq$ and $\otimes$ we have $a \leq b \implies \forall c\colon a + c \leq b + c$ which implies $b + c \leq \perp \implies a + c \leq \perp$. In the reverse direction, we have $b + b^{\perp} \leq \perp$ which, if we apply the hypothesis with $c \mapsto b^{\perp}$, gives us $b^{\perp} + a \leq \perp$. Because $(-)^{\perp}$ is dualizing, we then have $a \leq b$. □

Lemma 3.1

Proof. We show $F^\dagger$ is functorial: that $Ff = f\times f$ preserves $+$ and 0 follows from its componentwise definition, and that it is the same action in both components means it does commute with swap. So Ff is a valid $\mathsf{CMon}^\dagger$ morphism, and $F^\dagger$ preserves identities and composition in virtue of its componentwise definition.

We check that $\eta^\dagger$ is a natural transformation $1_{\mathsf{CMon}} \Rightarrow U^\dagger F^\dagger$. Given $\mathcal{X} := (X,+,0)$ and $\hat{\mathcal{X}} = (X^2,+^2,0^2)$, the mapping $x \mapsto (x,0)$ is a monoid homomorphism. We check naturality:

$$F^\dagger f(\eta^\dagger_{\mathcal{X}}(x)) = (f(x),0) = \eta^\dagger_{\mathcal{Y}}(f(x))$$

We check that $\varepsilon^\dagger$ is a natural transformation $F^\dagger U^\dagger \Rightarrow 1_{\mathsf{CMon}^\dagger}$. Given $\hat{\mathcal{Y}} := (Y^2,+^2,0^2,\sigma)$ and $\mathcal{Y} = (X,+,0,\dagger)$, the mapping $(y_1,y_2) \mapsto y_1 + y_2^\dagger$ sends $(0,0) \mapsto 0+0 = 0$, respects addition as $\varepsilon^\dagger_{\mathcal{Y}}(y_1,y_2) + \varepsilon^\dagger_{\mathcal{Y}}(z_1,z_2) = y_1 + y_2^\dagger + z_1 + z_2^\dagger = y_1 + z_1 + (y_2+z_2)^\dagger = \varepsilon^\dagger_{\mathcal{Y}}((y_1,y_2)+(z_1,z_2))$. And it respects involutions: $\varepsilon^\dagger_{\mathcal{Y}}(y_1,y_2)^\dagger = (y_1+y_2^\dagger)^\dagger = y_1^\dagger + y_2 = \varepsilon^\dagger_{\mathcal{Y}}((y_2,y_1)) = \varepsilon^\dagger_{\mathcal{Y}}((y_1,y_2)^\sigma)$. We check naturality:

$$\varepsilon^\dagger_{\mathcal{Y}}(F^\dagger f(x_1,x_2)) = \varepsilon^\dagger_{\mathcal{Y}}(f(x_1),f(x_2)) = f(x_1)+f(x_2)^\dagger = f(x_1+x_2^\dagger) = f(\varepsilon^\dagger_{\mathcal{X}}(x_1,x_2))$$

We check that $\eta_\dagger$ is a natural transformation $1_{\mathsf{CMon}^\dagger} \Rightarrow F^\dagger U^\dagger$. Given $\mathcal{X} := (X,+,0,\dagger)$ and $\hat{\mathcal{X}} = (X^2,+^2,0^2)$, the mapping $x \mapsto (x,x^\dagger)$ preserves the unit as $0 \mapsto (0,0^\dagger) = (0,0)$. Checking addition: $(x_1 + x_2)$ is sent to $(x_1+x_2,(x_1+x_2)^\dagger)$ which does equal $(x_1,x_1^\dagger)+(x_2,x_2^\dagger)$. The involution is also preserved: $\eta_{\dagger,\mathcal{X}}(x)^\sigma = (x,x^\dagger)^\sigma = (x^\dagger,x^{\dagger\dagger}) = \eta_{\dagger,\mathcal{X}}(x^\dagger)$. We check naturality:

$$F^\dagger f(\eta_{\dagger,\mathcal{X}}(x)) = (f(x),f(x^\dagger)) = (f(x),f(x)^\dagger) = \eta_{\dagger,\mathcal{Y}}(f(x))$$

We check that $\varepsilon_\dagger$ is a natural transformation $U^\dagger F^\dagger \Rightarrow 1_{\mathsf{CMon}}$. Given $\hat{\mathcal{Y}} := (Y^2,+^2,0^2,\sigma)$ and $\mathcal{Y} = (X,+,0)$, the mapping $(y_1,y_2) \mapsto y_1$ is clearly a monoid homomorphism. We check naturality:

$$\varepsilon_{\dagger,\mathcal{Y}}(F^\dagger f(x_1,x_2)) = \varepsilon_{\dagger,\mathcal{Y}}(f(x_1),f(x_2)) = f(x_1) = f(\varepsilon_{\dagger_{\mathcal{X}}}(x_1,x_2))$$

Now we verify the triangle identities for $\eta^\dagger$ and $\varepsilon^\dagger$.

$$\begin{array}{r}\varepsilon^\dagger_{F^\dagger\mathcal{X}}(F^\dagger(\eta^\dagger_{\mathcal{X}})(x_1,x_2))\\ \varepsilon^\dagger_{F^\dagger\mathcal{X}}((x_1,0),(x_2,0))\\ (x_1,0)+(x_2,0)^\sigma\\ (x_1,x_2)\end{array} \qquad \begin{array}{r}U^\dagger(\varepsilon^\dagger_{\mathcal{Y}})(\eta^\dagger_{U^\dagger\mathcal{Y}}(y))\\ \varepsilon^\dagger_{\mathcal{Y}}(y,0)\\ y+0\\ y\end{array}$$

Lastly we verify the triangle identities for $\eta_\dagger$ and $\varepsilon_\dagger$.

$$\begin{array}{r}\varepsilon_{\dagger,U^\dagger\mathcal{X}}(U^\dagger(\eta_{\dagger,\mathcal{X}})(x))\\ \varepsilon_{\dagger,U^\dagger\mathcal{X}}(x,x^\dagger)\\ x\end{array} \qquad \begin{array}{r}F^\dagger(\varepsilon_{\dagger_{\mathcal{Y}}})(\eta_{\dagger,F^\dagger\mathcal{Y}}(y_1,y_2))\\ F^\dagger(\varepsilon_{\dagger_{\mathcal{Y}}})((y_1,y_2)(y_2,y_1))\\ (\varepsilon_{\dagger_{\mathcal{Y}}}(y_1,y_2),\varepsilon_{\dagger_{\mathcal{Y}}}(y_2,y_1))\\ (y_1,y_2)\end{array}$$

$\square$

Lemma 3.2

Proof. Given any $\mathcal{Y} := (Y, +, 0, \leq) \in \mathsf{PreOrdCMon}$, the cartesian lift of a monoid morphism $f\colon \mathcal{X} \to U^{\leq}(\mathcal{Y})$ is given by the 'same' map f (because $U^{\leq}$ is faithful) from $\hat{\mathcal{X}}$ into $\mathcal{Y}$, where $\hat{\mathcal{X}}$ equips $\mathcal{X}$ with the pullback order: $x_1 \leq_{\mathcal{X}} x_2 := f(x_1) \leq_{\mathcal{Y}} f(x_2)$. First we need to establish that $\hat{\mathcal{X}}$ is a valid $\mathsf{PreOrdCMon}$ object: this means checking that $\leq$ is a preorder and that it is compatible with $+_{X}$. Its reflexivity and transitivity derive from the reflexivity and transitivity of $\leq_{\mathcal{Y}}$. For compatibility:

$$
\begin{array}{rr}
a \leq_{\mathcal{X}} b \wedge c \leq_{\mathcal{X}} d & \\
f(a) \leq_{\mathcal{Y}} f(b) \wedge f(c) \leq_{\mathcal{Y}} f(d) & \text{Defn of } \leq_{\mathcal{X}} \\
f(a) + f(c) \leq_{\mathcal{Y}} f(b) + f(d) & \mathcal{Y} \text{ has compatible } +,\ \leq \\
f(a+c) \leq_{\mathcal{Y}} f(b+d) & f \text{ is monoidal} \\
a + c \leq_{\mathcal{X}} b + d & \text{Defn of } \leq_{\mathcal{X}}
\end{array}
$$

That f is a valid $\mathsf{PreOrdCMon}$ morphism in addition to already being a CMon morphism just requires checking monotonicity, but it is monotone by the definition of $\leq_{\mathcal{X}}$.

Now that we have verified the proposed cartesian lift exists in $\mathsf{PreOrdCMon}$, we must test it has the required universal property by considering an arbitrary $\mathsf{PreOrdCMon}$ morphism $g\colon \mathcal{Z} \to \mathcal{Y}$ and a CMon morphism $w\colon U^{\perp}(\mathcal{Z}) \to \mathcal{X}$ such that $w \cdot f = g$ (note we use faithfulness to elide the difference between morphisms in CMon and $\mathsf{PreOrdCMon}$). Cartesianness means that w must moreover be a $\mathsf{PreOrdCMon}$ morphism. This requires that it additionally be monotone:

$$
\begin{array}{rr}
z_1 \leq_{\mathcal{Z}} z_2 & \\
g(z_1) \leq_{\mathcal{Y}} g(z_2) & g \text{ is monotone} \\
f(w(z_1)) \leq_{\mathcal{Y}} f(w(z_2)) & g = w \cdot f \\
w(z_1) \leq_{\mathcal{X}} w(z_2) & \text{Defn of } \leq_{\mathcal{X}}
\end{array}
$$

□

Lemma 3.3

Proof. Consider a GQ morphism $f\colon \mathcal{X} \to \mathcal{Y}$. We first check $U_{\pm}f$ is well-defined as a function. Let $R_{\mathcal{X}} \subseteq X^2$ be the reflexive elements of $U^{\otimes\neg\oplus}(\mathcal{X})$, i.e. $\{(a,b) \in X^2 \mid a \otimes b \leq \perp_{\mathcal{X}}\}$. Then $f(a) \otimes f(b) \leq f(\perp_{\mathcal{X}}) \leq \perp_{\mathcal{Y}}$ (using monotonicity and weak $\perp$ preservation of f as a GQ morphism), so $U^{\otimes\neg\oplus} f(a,b) = (f(a), f(b))$ is in $R_{\mathcal{Y}}$. $U_{\pm}f$ preserves $\perp$ because it is a restriction of $U^{\otimes\neg\oplus} f$, which preserves $\perp$. We must check the continuity condition for all sets of positions $A, B \subseteq \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]$. We need to show, from assuming $(A^{\perp} \cap \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]) \subseteq (B^{\perp} \cap \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}])$, that we can prove $(f(A)^{\perp} \cap \mathbb{N}[R_{\mathcal{Y}} + R_{\mathcal{Y}}]) \subseteq (f(B)^{\perp} \cap \mathbb{N}[R_{\mathcal{Y}} + R_{\mathcal{Y}}])$. We just need to show $A^{\perp} \subseteq B^{\perp}$, since then, by continuity of $U^{\otimes\neg\oplus} f$, we have $f(A)^{\perp} \subseteq f(B)^{\perp}$, which can be restricted to the intersection with $\mathbb{N}[R_{\mathcal{Y}} + R_{\mathcal{Y}}]$ to prove this goal.

Let (Γ, Δ) be an element $\mathbb{N}[X^2 + X^2]$, though not necessarily in $\mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]$. We can think of Γ as a multiset of pairs (of X elements) on the left and Δ as a multiset of pairs on the right. There exists a position

(Γ', Δ') with the same $\perp$-behavior which is in $\mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]$: we replace all left pairs $(\gamma_+, \gamma_-) \in X^2$ with $(\gamma_+, \gamma_+^\perp) \in R_{\mathcal{X}}$ and all the right pairs $(\delta_+, \delta_-) \in X^2$ with $(\delta_-^\perp, \delta_-) \in R_{\mathcal{X}}$. This has the same $\perp$ behavior because checking if $(\langle \mathtt{a}_+, \mathtt{a}_- \rangle, \langle \mathtt{b}_+, \mathtt{b}_- \rangle) \in \perp$ only depends on $\mathtt{a}_+$ and $\mathtt{b}_-$. This allows us to derive $A^\perp \subseteq B^\perp$:

$(\Gamma, \Delta) \in A^\perp$	
$(\Gamma', \Delta') \in A^\perp \cap \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]$	(Γ', Δ') has same incompatibilities
$(\Gamma', \Delta') \in B^\perp \cap \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]$	$(A^\perp \cap \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]) \subseteq (B^\perp \cap \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}])$ by hypothesis
$(\Gamma, \Delta) \in B^\perp$	(Γ, Δ) has same incompatibilities

Having shown that restricting the domain and codomain of morphisms $U^{\otimes\neg\oplus} f$ in $\mathsf{IF}_\pm$ are valid $\mathsf{IF}^{\mathsf{r}}_\pm$ morphisms, functoriality follows from functoriality of $U^{\otimes\neg\oplus}$. Next we show that $U_\pm$ is right adjoint to $F_\pm$. There is a natural bijection:

$$\mathsf{GQ}(F_\pm \mathcal{X},\ \mathcal{Q}) = \mathsf{GQ}(F^{\otimes\neg\oplus}(\iota^{\mathsf{r}} \mathcal{X}),\ \mathcal{Q}) \cong \mathsf{IF}_\pm(\iota^{\mathsf{r}} \mathcal{X},\ U^{\otimes\neg\oplus} \mathcal{Q}) \cong \mathsf{IF}^{\mathsf{r}}_\pm(\mathcal{X},\ U_\pm \mathcal{Q})$$

The first equality comes from $F_\pm = \iota^{\mathsf{r}} \cdot F^{\otimes\neg\oplus}$, and the first bijection is from $F^{\otimes\neg\oplus} \dashv U^{\otimes\neg\oplus}$. We establish the third natural bijection by first noting there is a natural transformation $\alpha\colon \iota^{\mathsf{r}} U_\pm \Rightarrow U^{\otimes\neg\oplus}$ whose components are the inclusion of reflexive elements into a general implication frame. Showing this inclusion function is continuous is tantamount to showing for $A, B \subseteq \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]$ and assuming $(A^\perp \cap \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}]) \subseteq (B^\perp \cap \mathbb{N}[R_{\mathcal{X}} + R_{\mathcal{X}}])$, that $A^\perp \subseteq B^\perp$. This follows from the same reasoning as above where every non-reflexive position has a corresponding reflexive position with the same $\perp$-behavior.

We can postcompose with these components to obtain a function $\mathsf{IF}^{\mathsf{r}}_\pm(\mathcal{X},\ U_\pm \mathcal{Q}) \rightarrowtail \mathsf{IF}_\pm(\iota^{\mathsf{r}} \mathcal{X}, U^{\otimes\neg\oplus} \mathcal{Q})$. In the other direction, we corestrict the function to reflexive elements. Thus our purported bijection is $\alpha_{\mathcal{Q}} \circ -$ and $(-)|_R$. These are inverse because corestriction and inclusion do not change how functions act on elements of the domain. □

Lemma 3.4

Proof. An element of $\mathbb{R}$ is an element $R \in \mathcal{P}[\mathbb{N}[X + X]]$ for which $R = \mathrm{RSR}(\mathrm{RSR}(R))$. An element $X \in \hat{\mathcal{X}}$ is a lower set of the involutive phase space $F^\otimes F^\neg(\mathcal{X})$ such that, in the quantale of lower sets, $X^{\perp\perp} = X$. The elements of $F^\otimes F^\neg(\mathcal{X})$ are in bijection with $\mathbb{N}[X + X]$, so its lower sets are a subset of $\mathcal{P}[\mathbb{N}[X + X]]$. First we must check that every element of $\mathbb{R}$ is a lower set. Suppose $x \leq y$ and $y \in R$ for some $R \in \mathbb{R}$. Because $\mathbb{R} = \mathrm{im}(\mathrm{RSR})$, there exists some $S \in \mathbb{N}[X + X]$ such that $R = \mathrm{RSR}(S)$. Then we infer that $\forall s \in S\colon s + y \in \perp$. However, $x \leq y$ means that anything which sums with y to be in $\perp$ must sum with x to be in $\perp$, therefore $\forall s \in S\colon s + x \in \perp$, which implies $x \in R$. So elements of $\mathbb{R}$ and $\hat{\mathcal{X}}$ are both lower sets, and we just need to verify that the property of being a fix point of RSR^2 is equivalent to being a fix point of $(-)^{\perp\perp}$. This is true because, as operations on lower sets, they are identical: $\mathrm{RSR}(X) = X^\perp = \{y \in \mathbb{N}[X + X] \mid \forall x \in X\colon x + y \in \perp\}$. □

Lemma 3.5

Proof. For $\mathtt{A}(\otimes \times ⅋)\mathtt{B}$ to be closed we need:

$\langle \mathtt{a}_+, \mathtt{a}_- \rangle (\otimes \times ⅋) \langle \mathtt{b}_+, \mathtt{b}_- \rangle \in \hat{\mathcal{X}}$	
$\langle \mathtt{a}_+ \otimes \mathtt{b}_+,\ \mathtt{a}_- ⅋ \mathtt{b}_- \rangle \in \hat{\mathcal{X}}$	Defn of $(\otimes \times ⅋)$
$\mathtt{a}_+ \otimes \mathtt{b}_+ \leq (\mathtt{a}_- ⅋ \mathtt{b}_-)^\perp$	Elementhood of $\hat{\mathcal{X}}$
$\mathtt{a}_+ \otimes \mathtt{b}_+ \leq \mathtt{a}_-^\perp \otimes \mathtt{b}_-^\perp$	Defn of ⅋

This follows from $\otimes$ being monotonic and $\mathtt{A}, \mathtt{B} \in \hat{\mathcal{X}}$. Now we check if $\mathtt{A}(\vee \times \wedge)\mathtt{B}$ is closed.

$\langle \mathtt{a}_+, \mathtt{a}_- \rangle (\vee \times \wedge) \langle \mathtt{b}_+, \mathtt{b}_- \rangle \in \hat{\mathcal{X}}$	
$\langle \mathtt{a}_+ \vee \mathtt{b}_+,\ \mathtt{a}_- \wedge \mathtt{b}_- \rangle \in \hat{\mathcal{X}}$	Defn of $(\vee \times \wedge)$
$\mathtt{a}_+ \vee \mathtt{b}_+ \leq (\mathtt{a}_- \wedge \mathtt{b}_-)^\perp$	Elementhood of $\hat{\mathcal{X}}$
$\mathtt{a}_+ \vee \mathtt{b}_+ \leq \mathtt{a}_-^\perp \vee \mathtt{b}_-^\perp$	Property of $(-)^\perp, \vee$

This follows from $\vee$ being monotonic and $\mathtt{A}, \mathtt{B} \in \hat{\mathcal{X}}$. Swapping elements is invariant for property of $\mathtt{a}_+ \otimes \mathtt{a}_- \leq \perp$ because $\otimes$ is commutative. Because we can express $\&_\tau$ and $⅋_\tau$ in terms of the swap, $\otimes_\tau$ and $\oplus_\tau$, these are also closed.

□

Lemma 4.1

Proof. We first show that $U^{\mathsf{c}}_\pm$ is a functor. We first check $U^{\mathsf{c}}_\pm$ sends $\mathsf{GQ}^{\mathsf{ji}}$ morphisms $f\colon \mathcal{Q} \to \mathcal{Q}'$ to $\mathsf{IF}^{\mathsf{c}}_\pm$ morphisms, which requires confirming that IC (idempotent and containment-satisfying) elements of $U^{\mathsf{c}}_\pm \mathcal{Q}$ are sent to IC elements of $U^{\mathsf{c}}_\pm \mathcal{Q}'$ by $U^{\mathsf{c}}_\pm f = f \times f$. Firstly, because f is a monoid homomorphism, it must send idempotent elements to idempotent elements. For any join-idempotent Girard quantale $\mathcal{Q}$, an element $\langle p, q \rangle$ of $U^{\mathsf{c}}_\pm \mathcal{Q}$ satisfies containment if $p \otimes q \otimes r \leq \perp$ for all $r \in \mathcal{Q}$, which is tantamount to $p \otimes q = 0$ (where 0 is the lattice bottom). So we must check that $f(p) \otimes f(q) = 0'_{\mathcal{Q}}$, which holds because 0 (the empty join) is preserved by f. Once again, $\perp$ is preserved by $U^{\mathsf{c}}_\pm f$ because it is the restriction of $U_\pm f$, which preserves $\perp$.

Let $C \subseteq X^2$ be the set of IC elements in $U^{\mathsf{c}}_\pm \mathcal{Q}$ for $\mathcal{Q} := (X, \otimes, \vee, \perp)$. Much like the proof of Lemma 3.3, we show continuity by using the continuity of $U_\pm f$. This requires us to prove, for any $A, B \subseteq \mathbb{N}[C+C]$ (with an arbitrary element of A being $\langle \Gamma, \Delta \rangle$ with $\Gamma, \Delta \in \mathbb{N}[C]$) and assuming that $(A^\perp \cap \mathbb{N}[C+C]) \subseteq (B^\perp \cap \mathbb{N}[C+C])$, that $A^\perp \subseteq B^\perp$. For any multiset $\Xi \in \mathbb{N}[X^2]$ of pairs of elements in $\mathcal{Q}$ in A, let $p_\Xi := \bigotimes_{(\xi_+, \xi_-) \in \Xi} \xi_+$ and $n_\Xi := \bigotimes_{(\xi_+, \xi_-) \in \Xi} \xi_-$.

Now, suppose $(\Theta, \Omega) \in A^\perp$, i.e. $\forall \langle \Gamma, \Delta \rangle \in A\colon\ p_\Gamma \otimes p_\Theta \otimes n_\Delta \otimes n_\Omega \leq \perp_{\mathcal{Q}}$. Because $\mathcal{Q}$ is join-idempotent, we have $p_\Theta = \bigvee_{e \in I \cap p_\Theta^\downarrow} e$ and $n_\Omega = \bigvee_{d \in I \cap n_\Omega^\downarrow} d$. Therefore, our hypothesis is $\bigvee_{e \in I \cap p_\Theta^\downarrow} \bigvee_{d \in I \cap n_\Omega^\downarrow} e \otimes d \otimes p_\Gamma \otimes n_\Delta \leq \perp_{\mathcal{Q}}$ and we need to prove, for an arbitrary $\langle \Gamma', \Delta' \rangle \in B$ that $\bigvee_{e \in I \cap p_\Theta^\downarrow} \bigvee_{d \in I \cap n_\Omega^\downarrow} e \otimes d \otimes p_{\Gamma'} \otimes n_{\Delta'} \leq \perp_{\mathcal{Q}}$. We can show this latter inequality by showing it for an arbitrary choice of $e \in I \cap p_\Theta^\downarrow$ and $d \in I \cap n_\Omega^\downarrow$.

Consider $\mathtt{x} = \langle \{(e, 0)\}, \{(0, d)\} \rangle$, which is in $\mathbb{N}[C+C]$ because e,d, and 0 are all idempotent elements of $\mathcal{Q}$ and because 0 is an absorbing element, so $e \otimes 0 = 0 \otimes d = 0$.

$e \otimes d \leq p_\Theta \otimes n_\Omega$	$e \leq p_\Theta \wedge d \leq n_\Omega$
$p_\Gamma \otimes e \otimes n_\Delta \otimes d \leq p_\Gamma \otimes p_\Theta \otimes n_\Delta \otimes n_\Omega$	$- \otimes (p_\Gamma \otimes n_\Delta)$ is monotone
$p_\Gamma \otimes e \otimes n_\Delta \otimes d \leq \bot_{\mathcal{X}}$	$(\Theta, \Omega) \in A^\perp$
$\mathtt{x} \in A^\perp$	
$\mathtt{x} \in B^\perp$	$(A^\perp \cap \mathbb{N}[C+C]) \subseteq (B^\perp \cap \mathbb{N}[C+C])$
$\forall \langle \Gamma', \Delta' \rangle : e \otimes d \otimes p_{\Gamma'} \otimes n_{\Theta'} \leq \bot_{\mathcal{X}}$	

Therefore this inequality holds for the join of all such e, d, which was needed to show $(\Theta, \Omega) \in B^\perp$.

The argument for extending the adjunction to $\mathsf{IF}^{\mathsf{c}}_{\pm}$ follows Lemma 3.3. There is a natural bijection $\mathsf{IF}^{\mathsf{r}}_{\pm}(\iota^{\mathsf{c}}\mathcal{X}, U_{\pm}\mathcal{Q}) \cong \mathsf{IF}^{\mathsf{c}}_{\pm}(\mathcal{X}, U^{\mathsf{c}}_{\pm}\mathcal{Q})$ given by restriction to IC elements and the inclusion of IC elements. □

Lemma 4.2

Proof. By construction, every element of $Q_\otimes$ is idempotent. We next need to show that joins are given by $\tilde{\bigvee}_{i \in I} x_i := \bigvee_{\varnothing \subset J \subseteq_{\text{fin}} I} \bigotimes_{j \in J} x_j$. First we must show this is idempotent:

$(\bigvee_{\varnothing \subset J \subseteq_{\text{fin}} I} \bigotimes_{j \in J} x_j) \otimes (\bigvee_{\varnothing \subset K \subseteq_{\text{fin}} I} \bigotimes_{k \in K} x_k)$	
$\bigvee_{\varnothing \subset J, K \subseteq_{\text{fin}} I} \bigotimes_{j \in J} x_j \otimes \bigotimes_{k \in K} x_k$	$\otimes$ distributes over joins
$\bigvee_{\varnothing \subset J \subseteq_{\text{fin}} I} \bigotimes_{j \in J} x_j$	See below

The last step is sound because for any choice of K, multiplying the product $\bigotimes_j X_j$ simply produces $\bigotimes_{j' \in J'} x_{j'}$ for some different J' in between $\varnothing$ and I.

To show this is an upper bound of x_i for $i \in I$, it is a join of all of the singleton sets $\{x_i\}$ and is therefore above them all in Q. To see it is the *least* upper bound when restricting to idempotent elements, let c be an idempotent element above all x_i. Using the monotonicity of $\otimes$ and $\leq$ we have, for any nonempty $J \subseteq I$, $\bigotimes_{j \in J} x_j \leq \bigotimes_{j \in J} c$, and the idempotency of c means $\bigotimes_{j \in J} x_i \leq c$. Therefore, because all elements in the join that constitutes $\tilde{\bigvee}_{i \in I} x_i$ are below c in $\mathcal{Q}$, we have $\tilde{\bigvee}_{i \in I} x_i \leq c$. Lastly, $\otimes$ distributes over this join because it is a join in Q. Note for the $I = \{1,2\}$ case, this formula evaluates to $x \mathbin{\tilde{\vee}} y := x \vee y \vee (x \otimes y)$. □

Lemma 4.3

Proof. Let $\hat{\mathcal{X}} = U^{\mathsf{c}}_{\pm}(\mathcal{Q})$ for some Girard quantale $\mathcal{Q}$. For $A(\otimes \times \tilde{\vee})B$ to be closed (i.e. preserve containment satisfaction), we need $\langle \mathtt{a}_+, \mathtt{a}_- \rangle (\otimes \times \tilde{\vee}) \langle \mathtt{b}_+, \mathtt{b}_- \rangle$ to satisfy containment, i.e. $(\mathtt{a}_+ \otimes \mathtt{b}_+) \otimes (\mathtt{a}_- \mathbin{\tilde{\vee}} \mathtt{b}_-)$ to equal $0_{\mathcal{Q}}$:

$$
\begin{array}{rl}
(\mathtt{a}_+ \otimes \mathtt{b}_+) \otimes (\mathtt{a}_- \mathbin{\tilde{\triangledown}} \mathtt{b}_-) & \\
(\mathtt{a}_+ \otimes \mathtt{b}_+) \otimes (\mathtt{a}_- \vee \mathtt{b}_- \vee (\mathtt{a}_- \otimes \mathtt{b}_-)) & \text{Defn of } \tilde{\triangledown} \\
(\mathtt{a}_+ \otimes \mathtt{b}_+ \otimes \mathtt{a}_-) \vee (\mathtt{a}_+ \otimes \mathtt{b}_+ \otimes \mathtt{b}_-) \vee (\mathtt{a}_+ \otimes \mathtt{b}_+ \otimes \mathtt{a}_- \otimes \mathtt{b}_-) & \otimes \text{ distributes over joins} \\
(0_{\mathcal{Q}} \otimes \mathtt{b}_+) \vee (\mathtt{a}_+ \otimes 0_{\mathcal{Q}}) \vee (0_{\mathcal{Q}} \otimes 0_{\mathcal{Q}}) & \langle \mathtt{a}_+, \mathtt{a}_- \rangle, \langle \mathtt{b}_+, \mathtt{b}_- \rangle \text{ satisfy containment} \\
0_{\mathcal{Q}} \vee 0_{\mathcal{Q}} \vee 0_{\mathcal{Q}} & 0_{\mathcal{Q}} \text{ is absorbing} \\
0_{\mathcal{Q}} & \vee \text{ is idempotent}
\end{array}
$$

□