Layered Monoidal Theories (talk proposal)

Leo Lobski
University College London, UK
leo.lobski.21@ucl.ac.uk

Fabio Zanasi
University College London, UK
f.zanasi@ucl.ac.uk

Categorical models often involve a translation between theories, for instance via an adjunction interpreting one theory in another, or a functor refining a more abstract theory by translating it to a more detailed one. Such translations are typically studied on a case-by-case basis, without a uniform mechanism allowing different theories to interact directly, e.g. by describing them within a single category. However, such translations arise naturally, since systems can be described at different *levels of abstraction*. The levels correspond to distinct perspectives on the system, emphasising different details or features. It is common to regard some levels as *coarser* and others as more *fine-grained*, with translations corresponding intuitively to “zooming in” or “abstracting away” the details. For example, reaction networks treat chemical compounds as placeholders with no internal structure, whereas the molecular structure is considered when modelling the compounds as graphs [1]. Similarly, higher level descriptions in computer architecture must ultimately be implemented as microelectronic circuits [5]. The left side of Figure 1 illustrates a translation from a coarse layer, consisting of a restricted set of English names for molecules, to a finer layer of molecular graphs equipped with rewriting rules. These examples demonstrate the ubiquity of mixed levels of abstraction across sciences: our aim is, therefore, to provide a general formalism for mathematical reasoning across different layers.

In this talk, we introduce *layered monoidal theories* [6, 7], a mathematical framework where layers of abstractions for a wide range of phenomena may be studied algebraically within the same string diagrammatic formalism. Intuitively, a layered monoidal theory can be thought of as a “glueing” of different monoidal theories, whose semantics is given by an indexed monoidal category [7].

We show a typical term (morphism) in a layered monoidal theory in Figure 2a. Here x is a morphism in the category ω (drawn as a box on the left) and y is a morphism in the category τ (drawn as a box on the right). The dotted lines indicate that x and y sit above ω and τ, and that the “functor boundary” $\blacktriangleleft_f$ sits above f – these are formally not part of the term and are included for illustrative purposes only. The morphism x can be, moreover, pushed through the functor boundary, as shown in Figure 2b. Moreover, we allow functor boundaries $\triangleright_f$ in the opposite direction, yielding diagrams of the shape shown in Figure 2c called *windows*. Windows allow one to “peek in” at the semantics of the functor f – without performing the full translation. Furthermore, in many scenarios, one encounters transformations

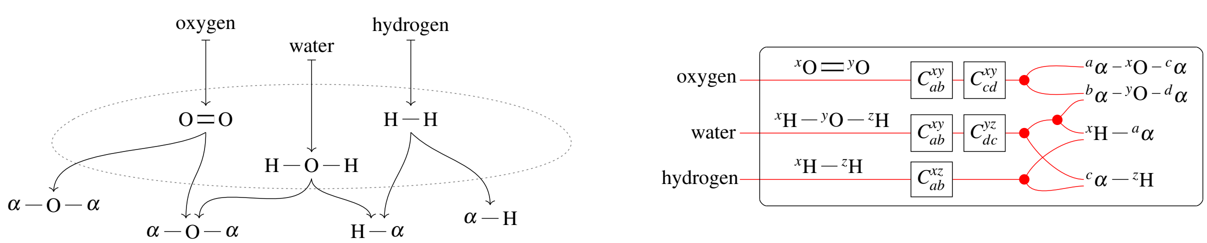
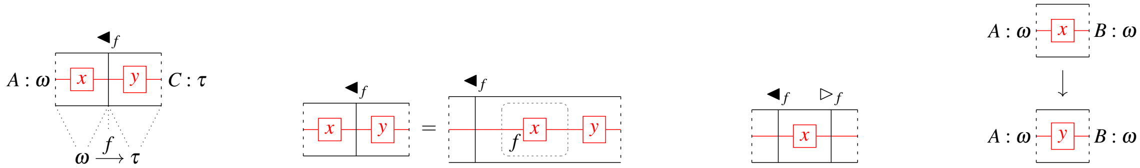


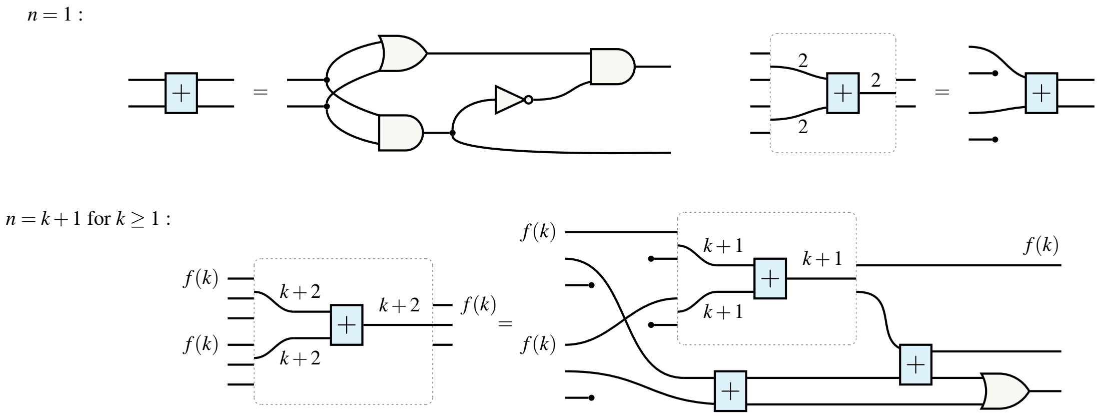
Figure 1: Left: an informal translation from a coarser language (English names) to a finer language (molecular graphs). Right: A formalisation of said translation as a term in a layered monoidal theory.

Submitted to:
ACT 2026

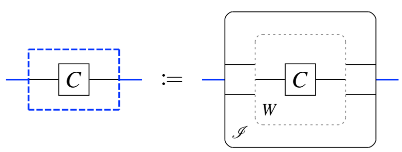


(a) A typical term. (b) An equation. (c) A window. (d) A 2-cell.

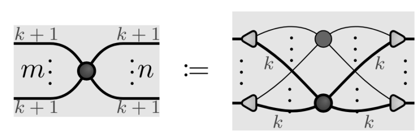
Figure 2: Various features of layered monoidal theories.



(a) The recursive definition of addition of binary numbers in the layered theory of digital circuits.



(b) Definition of the impedance box of [2] in the layered theory of electrical circuits.



(c) Recursive definition of a spider in the scalable notation for linear algebra from [3].

Figure 3: Instances of layered monoidal theories.

between processes that are non-trivial, in the sense that they are not identities, thus genuinely modifying the process. Within layered theories, such transformations are modelled as *2-cells*, drawn in Figure 2d.

After introducing the syntax of layered monoidal theories and briefly sketching their semantics, we illustrate how existing constructions are captured within the framework in a systematic way. This is done via three examples: (1) operations that act on "bundles" of wires of arbitrary bit width in digital circuits [5] (Figure 3a), (2) impedance boxes [2] that "mix" the electrical circuit components with their interpretation in graphical affine algebra (Figure 3b), and (3) the scalable notation for the ZX-calculus [4] and graphical symplectic algebra [3] which compactly represents quantum (and symplectic) processes at a large scale (Figure 3c).

References

[1] Jakob L. Andersen, Christoph Flamm, Daniel Merkle & Peter F. Stadler (2019): *Chemical Transformation Motifs – Modelling Pathways as Integer Hyperflows*. *IEEE/ACM transactions on computational biology and*

bioinformatics 16(2), pp. 510–523, doi:10.1109/TCBB.2017.2781724.

[2] Guillaume Boisseau & Paweł Sobociński (2021): *String Diagrammatic Electrical Circuit Theory*, doi:10.48550/ARXIV.2106.07763.

[3] Robert I. Booth, Titouan Carette & Cole Comfort (2024): *Graphical Symplectic Algebra*. arXiv:2401.07914.

[4] Titouan Carette, Dominic Horsman & Simon Perdrix (2019): *SZX-Calculus: Scalable Graphical Quantum Reasoning*. 138, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, pp. 55:1–55:15, doi:10.4230/LIPICS.MFCS.2019.55.

[5] George Kaye (2025): *Foundations of Digital Circuits: Denotation, Operational, and Algebraic Semantics*, doi:10.48550/arXiv.2502.08497.

[6] Leo Lobski & Fabio Zanasi (2026): *Layered Monoidal Theories I: Diagrammatic Algebra and Applications*. arXiv:2602.19776.

[7] Leo Lobski & Fabio Zanasi (2026): *Layered Monoidal Theories II: Fibrational Semantics*. arXiv:2602.22373.