Temporal sheaf theory for reconciling temporal complexity within public health modeling

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In this paper, we develop temporal sheaf theory with applications to public health modeling. We explicate discrete temporal sheaves in terms of a category of discrete subunit-intervals $\mathbf{Z}$. We characterize the topos-theoretic properties of the category $[\mathbf{Z}^{\mathsf{op}}, [\mathcal{C}, \mathbf{Set}]]$ of discrete $[\mathcal{C}, \mathbf{Set}]$-narratives for small categories $\mathcal{C}$ and interpret the logical structure of its subobject classifier to describe temporal properties of data. When $\mathcal{C}$ is the walking parallel arrows category, $[\mathcal{C}, \mathbf{Set}]$ is the category of directed graphs, making these discrete $[\mathcal{C}, \mathbf{Set}]$-narratives dynamic analogues of graphs, among which we find dynamic analogues of paths and cliques. We ground the formulations advanced here with concrete implementations in Catlab.jl. We conclude our paper by discussing applications of temporal sheaf theory to temporally complex public health concerns. In particular, for public health agent-based models (ABMs), it is crucial to reason about which individuals persist in a population over time intervals or are present within a time period, how individuals qualify for observation, and when individuals simultaneously experience combinations of conditions. We discuss how temporal paths can produce insights for contact-tracing and how using temporal sheaves can both improve the understanding of communicable diseases and reasoning about individual-level, public health data or ABM-generated data.

1 Introduction/Background

In public health, understanding health trends and the dynamics induced by health policies—such as those involving pathogen spread (see Figure 1) or elevating vaccination rates—can be challenging. Asking what individuals persist in a population over time, how these individuals qualify for observation, and when they may have overlapping conditions benefits from robust, mathematically transparent, methodological approaches. To understand such effects, analysts who assist policymakers investigate time-series data and simulation models to generate insights from both localized, individual-level information as well as global population perspectives.

To manage the overlapping temporal complexity present in analysis of individual-level epidemiological data (e.g., outbreak line lists) and states of agent-based models (ABMs), in this paper, we expand on the sheaf-theoretic techniques introduced by Bumpus et al. in [1] to model time-varying data. Focusing on the discrete-time case, we consider (co)presheaves on a posetal

*Co-funded by the European Union and Estonian Research Council through the Mobilitas 3.0 (MOB3JD1227).

Submitted to ACT 2026.

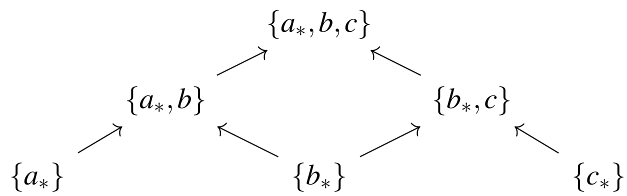

Figure 1: Example of pathogen spread as a cosheaf: a, b, c are persons, an asterisk subscript $_*$ denotes infection, and arrows send each person to (an infected or uninfected version of) themselves. Here, a_* interacts with b, resulting in b_*. Then, a_* loses contact with b_*, but b_* makes contact with c, infecting c.

category of discrete subunit-intervals $\mathbf{Z}$ as discrete temporal sheaves. We leverage the propositional calculus of subobject lattices for the topos of these discrete temporal sheaves to provide a language for time-based queries on both set-valued and presheaf-valued data, and we implement examples using Catlab.jl. We conclude this paper with vignettes of applications to public health, with special attention to ABMs, as they are well-suited to simulating populations, individuals, and interactions among persons and their environment.

1.1 Acknowledgments

The authors express their gratitude to the Pacific Institute for Mathematical Sciences (PIMS) and Banff International Research Station (BIRS) for funding the "PIMS-BIRS Teamup!" workshop (award 26frg502) that made this work possible, as well as the insights offered by Kris Brown, Wilmer Leal, and Evan Patterson. We also greatly appreciate the opportunity for extensive discussion with Juxin Liu, who contributed much to our understanding.

2 Temporal Sheaf Theory

We review the (co)sheaves of time-varying data introduced in [1], which we will call *temporal (co)sheaves*, and we expand on their categorical properties.

2.1 Introducing Temporal Sheaves

In defining temporal sheaves, we follow [1] while adjusting notations for our purposes. We begin by specializing [2] to the case of total orders representing time.

Definition 2.1. Let $(\tau, \leq)$ be a totally ordered set: we call it our *timeline* and its elements *times*. Let $(\mathbf{I}_\tau, \subseteq)$ be the poset of *(closed) intervals* of τ of the form $[a,b]$ for $a, b \in \tau$ with $a \leq b$, ordered by containment $\subseteq$, so that $[a_1, b_1] \subseteq [a_2, b_2]$ if $a_2 \leq a_1 \leq b_1 \leq b_2$.

The *Johnstone coverage* on the posetal category $\mathbf{I}_\tau$ is comprised of the families $\{[a,b]\}$ and $\{[a,p],[p,b]\}$ covering $[a,b] \in \mathbf{I}_\tau$ for all $p \in \tau$ satisfying $a \leq p \leq b$.[1] The coverage induces a Grothendieck topology $\mathcal{J}_\tau$ on $\mathbf{I}_\tau$. For a category $\mathbf{D}$ with pullbacks, we call the category of $\mathbf{D}$-valued sheaves on the site $(\mathbf{I}_\tau, \mathcal{J}_\tau)$ the *category of persistent* $\mathbf{D}$*-narratives*, denoted $\mathbf{Pe}_\tau(\mathbf{D})$; and if $\mathbf{D}$ has pushouts, we call the category of $\mathbf{D}$-valued cosheaves on the site $(\mathbf{I}_\tau, \mathcal{J}_\tau)$ the *category of cumulative* $\mathbf{D}$*-narratives*, denoted $\mathbf{Cu}_\tau(\mathbf{D})$. ♢

[1]We verify that this indeed constitutes a coverage in the appendix.

Unraveling Definition 2.1, we find that a persistent $\mathbf{D}$-narrative is a functor $X\colon \mathbf{I}_\tau^{\mathsf{op}} \to \mathbf{D}$ such that for all $a,p,b \in \tau$ with $a \leq p \leq b$, we have the following pullback square given by the restriction maps[2] of X:

$$\begin{array}{ccccc} & & X[a,b] & & \\ & \swarrow & \lrcorner & \searrow & \\ X[a,p] & & & & X[p,b] \\ & \searrow & & \swarrow & \\ & & X[p,p] & & \end{array}$$

where the interval $[p,p]$ is the only interval in $\mathbf{I}_\tau$ contained in both elements of the cover $\{[a,p],[p,b]\}$. When $\mathbf{D} \coloneqq \mathbf{Set}$, semantically, for an interval $I \in \mathbf{I}_\tau$, we interpret $x \in X(I)$ as "x persists in X throughout the time interval I." Given $J \in \mathbf{I}_\tau$ with $J \subseteq I$, we interpret the induced restriction function $-\big|_J\colon X(I) \to X(J)$ as "$x\big|_J$ refers to what x is, in X, throughout the subinterval $J \subseteq I$"; we can think of $x\big|_J$ as a "snapshot" of x over the shorter interval J. Notably, different sections in $X(I)$ may have been equal during a smaller interval $J \subseteq I$. The pullback condition then asserts that "what persists in X throughout $[a,b]$ are precisely the pairs (x,y) where x persists throughout $[a,p]$, y persists throughout $[p,b]$, and x and y are the same at time p." The persistent behavior across the interval $[a,b]$ is entirely determined by the persistent behaviors at $[a,p]$ and $[p,b]$ and their compatibility at p for any $p \in [a,b]$.

Cumulative $\mathbf{D}$-narratives are functors $Y\colon \mathbf{I}_\tau \to \mathbf{D}$ satisfying the dual pushout condition. When $\mathbf{D} \coloneqq \mathbf{Set}$, semantically, for $I \in \mathbf{I}_\tau$, we interpret $y \in Y(I)$ as "y exists in Y sometime in the interval I." Given $J \in \mathbf{I}_\tau$ with $I \subseteq J$, we interpret the induced extension function $-\big|^J\colon X(I) \to X(J)$ as "$y\big|^J$ is what y is in Y in the superinterval $J \supseteq I$"; we think of $y\big|^J$ as "subsuming" y. Notably, different sections in $Y(I)$ may merge sometime in a larger interval $J \supseteq I$. The pushout condition then asserts that "what exists in X sometime in the interval $[a,b]$ consists of what exists sometime in $[a,p]$ and what exists sometime in $[p,b]$, counting each thing that exists at time p only once." The cumulative behavior in the interval $[a,b]$ is again determined by the cumulative behaviors at $[a,p]$ and $[p,b]$ and their compatibility at p for any $p \in [a,b]$.

Typically we take τ to be a closed interval of $\mathbb{R}$ or $\mathbb{Z}$. In particular, we call (co)sheaves on $\mathbf{I}_\mathbb{R}$ *continuous* narratives and (co)sheaves on $\mathbf{I}_\mathbb{Z}$ *discrete* narratives. In the latter case, such narratives turn out to be equivalent to presheaves on the following category.

Definition 2.2. Let $(\mathbf{Z}, \subseteq)$ be the induced subposet of $(\mathbf{I}_\mathbb{Z}, \subseteq)$ spanned by intervals of length at most one.[3] More concretely, $\mathbf{Z}$ is the posetal category consisting of:

- objects $[t] \coloneqq [t,t]$ and $[t,t+1]$ for $t \in \mathbb{Z}$;
- non-identity morphisms $[t] \to [t,t+1]$ and $[t+1] \to [t,t+1]$ for $t \in \mathbb{Z}$. ◊

We call $\mathbf{Z}$ the *category of discrete subunit-intervals.*

Proposition 2.3. *The category $\mathbf{Pe}_\mathbb{Z}(\mathbf{D})$ of persistent discrete $\mathbf{D}$-narratives is equivalent to the functor category $[\mathbf{Z}^{\mathsf{op}}, \mathbf{D}]$ of presheaves on $\mathbf{Z}$. The category $\mathbf{Cu}_\mathbb{Z}(\mathbf{D})$ of cumulative discrete $\mathbf{D}$-narratives is equivalent to the functor category $[\mathbf{Z}, \mathbf{D}]$ of copresheaves on $\mathbf{Z}$. Both these equivalences are given by restriction along the inclusion $\mathbf{Z} \hookrightarrow \mathbf{I}_\mathbb{Z}$ in one direction and iterated pullbacks or pushouts in the other.*

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[2]We call the morphism obtained by applying a (pre)sheaf (like the functors $\mathbf{I}_\tau^{\mathsf{op}} \to \mathbf{D}$ in $\mathbf{Pe}_\tau(\mathbf{D})$) to a morphism in the underlying category (like $\mathbf{I}_\tau$) a *restriction map.* Dually, we call the morphism obtained by applying a co(pre)sheaf (like the functors $\mathbf{I}_\tau \to \mathbf{D}$ in $\mathbf{Cu}_\tau(\mathbf{D})$) to a morphism in the underlying category (like $\mathbf{I}_\tau$) an *extension map.*

[3]The letter $\mathbf{Z}$ is chosen to be reminiscent of $\mathbb{Z}$ as well as to evoke the zigzag shape of the category.

We prove this proposition—and others throughout the text—in Appendix A.

The functor categories $[\mathbf{Z},\mathbf{D}]$ and $[\mathbf{Z}^{\mathsf{op}},\mathbf{D}]$ are (non-canonically) isomorphic as $\mathbf{Z}$ and $\mathbf{Z}^{\mathsf{op}}$ are.

Proposition 2.4. *There is an isomorphism* $\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}^{\mathsf{op}}$ *that sends* $[t] \mapsto [t,t+1]$ *and* $[t,t+1] \mapsto [t+1]$ *for all* $t \in \mathbb{Z}$.

By Proposition 2.4, any categorical result on $[\mathbf{Z}^{\mathsf{op}},\mathbf{D}]$ transfers to $[\mathbf{Z},\mathbf{D}]$, and we may elide the difference between persistent and cumulative discrete narratives as well as between presheaves on $\mathbf{Z}$ and sheaves on $(\mathbf{I}_{\mathbb{Z}},\mathcal{J}_{\mathbb{Z}})$.

The category $\mathbf{Z}$ is actually fibered over a more familiar category.

Definition 2.5. Let $\mathcal{G}$ denote the walking parallel arrows category, consisting of:

- two objects, E and V;
- two nonidentity morphisms, $\mathsf{s},\mathsf{t}\colon \mathsf{E} \rightrightarrows \mathsf{V}$.

Then let $\mathbf{Grph} := [\mathcal{G},\mathbf{Set}]$ denote the standard category of (directed) graphs. ◊

Proposition 2.6. *The category* $\mathbf{Z}^{\mathsf{op}}$ *is isomorphic to the category of elements of the doubly-infinite path graph*

$$\cdots \xrightarrow{-3} -2 \xrightarrow{-2} -1 \xrightarrow{-1} 0 \xrightarrow{0} 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots,$$

viewed as a functor $\mathcal{P}_{\mathbb{Z}}\colon \mathcal{G} \to \mathbf{Set}$ *given by* $\mathcal{P}_{\mathbb{Z}}(\mathsf{E}) := \mathcal{P}_{\mathbb{Z}}(\mathsf{V}) := \mathbb{Z}$, $\mathcal{P}_{\mathbb{Z}}(\mathsf{s}) := \mathrm{id}_{\mathbb{Z}}$, *and* $\mathcal{P}_{\mathbb{Z}}(\mathsf{t}) := \mathrm{succ}$ *with* $\mathrm{succ}(t) := t+1$. *Consequently, there is a discrete opfibration* $\mathbf{Z}^{\mathsf{op}} \to \mathcal{G}$ *sending* $[t] \to [t,t+1]$ *to* $\mathsf{s}\colon \mathsf{E} \to \mathsf{V}$ *and* $[t+1] \to [t,t+1]$ *to* $\mathsf{t}\colon \mathsf{E} \to \mathsf{V}$ *for all* $t \in \mathbb{Z}$.

Corollary 2.7. *The category* $[\mathbf{Z}^{\mathsf{op}},\mathbf{Set}]$ *is equivalent to the slice category* $\mathbf{Grph}/\mathcal{P}_{\mathbb{Z}}$.

```julia
# Define Z with 2 time points
@present Z2(FreeSchema) begin
  (t₀, t₁)::Ob
  (t₀_to_t₁)::Ob
  t₀_incl::Hom(t₀, t₀_to_t₁)
  t₁_incl::Hom(t₁, t₀_to_t₁)
end;

# Opposite of Z
Zop2 = op(Z2)

# Instantiate it on an acset
@acset_type Zop2Inst(Zop2)

# Compute the representable at the apex
h = representable(Zop2Inst, :t₀_to_t₁)

# Display via category of elements
to_graphviz(elements(h))
```

Listing 1: We cannot define the entire infinite category $\mathbf{Z}^{\mathsf{op}}$ in Catlab.jl, but we define here a subcategory of $\mathbf{Z}^{\mathsf{op}}$ spanned by intervals in $[0,1]$.

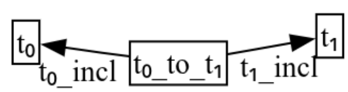


Figure 2: Depicting the subcategory of $\mathbf{Z}^{\mathsf{op}}$ spanned by intervals in $[0,1]$.

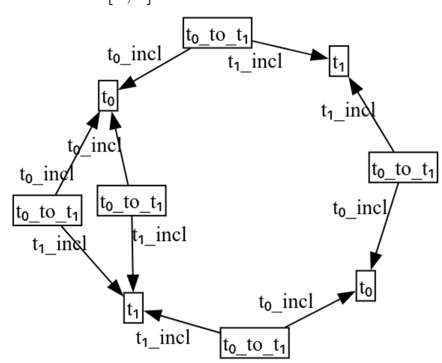


Figure 3: Depicting the category of elements of the subobject classifier for the subcategory of $\mathbf{Z}^{\mathsf{op}}$ spanned by intervals in $[0,1]$.

So a discrete $\mathbf{Set}$-narrative $X\colon \mathbf{Z}^{\mathsf{op}} \to \mathbf{Set}$ is equivalently a graph $G_X\colon \mathcal{G} \to \mathbf{Set}$ equipped with a morphism to $\mathcal{P}_{\mathbb{Z}}$. Concretely, G_X has vertex-set $\sum_{t\in\mathbb{Z}} X[t]$ with its canonical projection to the vertex-set $\mathbb{Z}$ of $\mathcal{P}_{\mathbb{Z}}$; edge-set $\sum_{t\in\mathbb{Z}} X[t,t+1]$ with its canonical projection to the edge-set $\mathbb{Z}$ of $\mathcal{P}_{\mathbb{Z}}$; and source (resp. target) function given by the coproduct of restriction maps $X[t,t+1] \to X[t]$

(resp. $X[t,t+1] \to X[t+1]$) over $t \in \mathbb{Z}$. It is a graph whose vertices are labeled with integers and whose edges each have a source vertex labeled t and a target vertex labeled $t+1$ for some $t \in \mathbb{Z}$. We sometimes "draw" a discrete **Set**-narrative as exactly such a labeled graph. Furthermore, recall that Proposition 2.4 ensures that $X\colon \mathbf{Z}^{\mathsf{op}} \to \mathbf{Set}$ is equivalent to a sheaf $\overline{X}\colon \mathbf{I}_{\mathbb{Z}}^{\mathsf{op}} \to \mathbf{Set}$. The pullback condition of the sheaf implies that it sends $[t,t+d]$ to the set of length-d paths in G_X that originate from vertices labeled t for $t,d \in \mathbb{Z}$.

Returning to arbitrary timelines τ, in the case of $\mathbf{D} := \mathbf{Set}$, the category $\mathbf{Pe}_\tau(\mathbf{Set})$ of persistent **Set**-narratives is a Grothendieck topos (as is $\mathbf{Cu}_{\mathbb{Z}}(\mathbf{Set}) \simeq [\mathbf{Z}, \mathbf{Set}]$). We may characterize when maps $\tau \to \tau'$ induce geometric morphisms $\mathbf{Pe}_{\tau'}(\mathbf{Set}) \to \mathbf{Pe}_\tau(\mathbf{Set})$, yielding structure-preserving maps between narratives at different timescales.

Proposition 2.8. *Let $r\colon \tau \to \tau'$ be a monotone map. Then composition with its induced map $\mathbf{I}_\tau \to \mathbf{I}_{\tau'}$ yields a geometric morphism $\mathbf{Pe}_{\tau'}(\mathbf{Set}) \to \mathbf{Pe}_\tau(\mathbf{Set})$ if and only if r satisfies the following condition: for all $c' \in \tau'$, there exist $a,b \in \tau$ with $r(a) \le c' \le r(b)$.*

Example 2.9. The inclusion $\iota\colon \mathbb{Z} \hookrightarrow \mathbb{R}$ induces an inclusion $\iota\colon \mathbf{I}_{\mathbb{Z}} \hookrightarrow \mathbf{I}_{\mathbb{R}}$, which in turn induces a geometric morphism $\mathbf{Pe}_{\mathbb{R}}(\mathbf{Set}) \to \mathbf{Pe}_{\mathbb{Z}}(\mathbf{Set})$ given by restriction along $\iota\colon \mathbf{I}_{\mathbb{Z}} \hookrightarrow \mathbf{I}_{\mathbb{R}}$. We think of this as sampling a continuous narrative at countably many timestamps that tend toward $-\infty$ in one direction and $+\infty$ in the other. ◊

Of particular interest among persistent **Set**-narratives are the contravariant representable functors $h^{[a,b]} := \mathbf{I}_\tau(-,[a,b])\colon \mathbf{I}_\tau^{\mathsf{op}} \to \mathbf{Set}$ for $[a,b] \in \mathbf{I}_\tau$. Note that, since $\mathbf{I}_\tau$ is posetal, the representables only take the values $\varnothing$ or the singleton set $*$.

Proposition 2.10. *Every representable presheaf on $\mathbf{I}_\tau$ is a sheaf in $\mathbf{Pe}_\tau(\mathbf{Set})$.*

When $\tau := \mathbb{Z}$, we may consider the representable presheaves on $\mathbf{Z}$. For $t \in \mathbb{Z}$, the functor $h^{[t]} := \mathbf{Z}(-,[t])\colon \mathbf{Z}^{\mathsf{op}} \to \mathbf{Set}$ is given on objects $I \in \mathbf{Z}$ by

$$h^{[t]}(I) := \begin{cases} * & \text{if } I = [t] \\ \varnothing & \text{otherwise,} \end{cases}$$

where $*$ is the singleton set. We may depict the functor $h^{[t]}$ as the following zigzag diagram, with integer labels along the bottom, indicating the value of $h^{[t]}([t-1])$ directly above the label $t-1$, the value of $h^{[t]}([t-1,t])$ between $t-1$ and t, the value of $h^{[t]}[t]$ directly above t, etc. (often we omit the ellipses when every unwritten entry is $\varnothing$).

$$\begin{array}{ccccccccc} & \cdots & & \varnothing & & \varnothing & & \cdots & \\ & & \swarrow & & \searrow \; \swarrow & & \searrow & & \\ \cdots & & \varnothing & & * & & \varnothing & & \cdots \\ \\ \cdots & & t-1 & & t & & t+1 & & \cdots \end{array}$$

As a labeled graph, $h^{[t]}$ is a single t-labeled vertex.

Meanwhile $h^{[t,t+1]} := \mathbf{Z}(-,[t,t+1])\colon \mathbf{Z}^{\mathsf{op}} \to \mathbf{Set}$ is given on $I \in \mathbf{Z}$ by

$$h^{[t,t+1]}(I) := \begin{cases} * & \text{if } I \in \{[t],[t+1],[t,t+1]\}, \text{ i.e. if } I \subseteq [t,t+1] \\ \varnothing & \text{otherwise.} \end{cases}$$

We depict $h^{[t,t+1]}$ like so; as a labeled graph, it is an edge from a vertex labeled t to a vertex labeled $t+1$.

$$\begin{array}{ccccccccc} \varnothing & & & & * & & & & \varnothing \\ & \searrow & & \swarrow & & \searrow & & \swarrow & \\ & & * & & & & * & & \\ & & & & & & & & \\ & & t & & & & t+1 & & \end{array}$$

Note that a covariant representable functor $h_{[a,b]} := \mathbf{I}_\tau([a,b],-)\colon \mathbf{I}_\tau \to \mathbf{Set}$ is *not* necessarily a cosheaf in $\mathbf{Cu}_\tau(\mathbf{Set})$. For instance, for $\tau := \mathbb{Z}$, the representable functor $h_{[-1,1]}\colon \mathbf{I}_\mathbb{Z} \to \mathbf{Set}$ is *not* a cosheaf because the following square is *not* a pushout:

$$\begin{array}{ccccc} & & h_{[-1,1]}[-1,1] \cong * & & \\ & \nearrow & & \nwarrow & \\ h_{[-1,1]}[-1,0] \cong \varnothing & & & & h_{[-1,1]}[0,1] \cong \varnothing \\ & \nwarrow & & \nearrow & \\ & & h_{[-1,1]}[0,0] \cong \varnothing & & \end{array}$$

While we introduced $\mathbf{D}$-narratives for an arbitrary category $\mathbf{D}$, we will typically consider the case where $\mathbf{D} := [\mathcal{C}, \mathbf{Set}]$ for a small category $\mathcal{C}$. This includes cases like $\mathcal{C} := \mathcal{G}$, so that $[\mathbf{Z}^{\mathsf{op}}, [\mathcal{G}, \mathbf{Set}]] \cong [\mathbf{Z}^{\mathsf{op}}, \mathbf{Grph}]$ becomes the category of discrete $\mathbf{Grph}$-narratives, modeling time-varying graphs; or other values of $\mathcal{C}$ that may parametrize *spatial* data. Then the category of discrete $[\mathcal{C}, \mathbf{Set}]$-narratives is

$$[\mathbf{Z}^{\mathsf{op}}, [\mathcal{C}, \mathbf{Set}]] \cong [\mathbf{Z}^{\mathsf{op}} \times \mathcal{C}, \mathbf{Set}],$$

to which we apply the theory of set-valued presheaves, including the Yoneda Lemma and topos theory.

Lemma 2.11 (Yoneda)**.** *Given a discrete $[\mathcal{C}, \mathbf{Set}]$-narrative $F\colon \mathbf{Z}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$ and $(I,c) \in \mathbf{Z}^{\mathsf{op}} \times \mathcal{C}$, we have an isomorphism*[4]

$$F(I,c) \cong [\mathbf{Z}^{\mathsf{op}} \times \mathcal{C}, \mathbf{Set}](\mathbf{Z}(-,I) \times \mathcal{C}(c,=), F)$$

natural in F, I, and c.

Proposition 2.12. *The category $[\mathbf{Z}^{\mathsf{op}}, [\mathcal{C}, \mathbf{Set}]]$ of discrete $[\mathcal{C}, \mathbf{Set}]$-narratives is a Grothendieck (and thus elementary) topos.*[5]

So we study the category $[\mathbf{Z}^{\mathsf{op}}, [\mathcal{C}, \mathbf{Set}]]$ as a topos, following Mac Lane and Moerdijk in [3].

2.2 Discrete Narrative Topoi

We proceed by investigating the topos-theoretic properties of our category of discrete $[\mathcal{C}, \mathbf{Set}]$-narratives.

[4]Here $\mathbf{Z}(-,I) = h_I$ is a contravariant representable functor $\mathbf{Z}^{\mathsf{op}} \to \mathbf{Set}$ while $\mathcal{C}(c,=)$ is a covariant representable functor $\mathcal{C} \to \mathbf{Set}$, so their product is a functor $\mathbf{Z}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$.

[5]Every Grothendieck topos (i.e., category of set-valued sheaves on a site) is an elementary topos. An elementary topos is a category that has finite limits (here given objectwise), is cartesian closed, and has a subobject classifier. We will investigate the latter two properties shortly.

2.2.1 Cartesian Closure

Like all presheaf categories, the category of discrete $[\mathcal{C},\mathbf{Set}]$-narratives (that is, $X : \mathbf{Z}^{\mathsf{op}} \to [\mathcal{C},\mathbf{Set}]$) has products given objectwise and is Cartesian closed. In particular, for $F,G\colon \mathbf{Z}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$, their Cartesian closure G^F is another discrete narrative given as follows. For ease of notation, for $I,J \in \mathbf{Z}$, we write $F_I(c) := F(I,c)$ and $G_J(c) := G(J,c)$, so that $F_I, G_J\colon \mathcal{C} \to \mathbf{Set}$. We also write $G_J^{F_I}$ for the cartesian closure of F_I and G_J in $[\mathcal{C},\mathbf{Set}]$.

Proposition 2.13. *For $t \in \mathbb{Z}$ and $c \in \mathcal{C}$, the cartesian closure $G^F : \mathbf{Z}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$ sends*[6]

$$([t],c) \mapsto G_{[t]}^{F_{[t]}}(c)$$

and $([t,t+1],c)$ to the limit of the diagram

$$G_{[t]}^{F_{[t]}}(c) \rightarrow G_{[t]}^{F_{[t,t+1]}}(c) \leftarrow G_{[t,t+1]}^{F_{[t,t+1]}}(c) \rightarrow G_{[t+1]}^{F_{[t,t+1]}}(c) \leftarrow G_{[t+1]}^{F_{[t+1]}}(c)$$

where the outer maps are induced by the pair of restriction maps $F_{[t]} \leftarrow F_{[t,t+1]} \to F_{[t+1]}$ while the inner maps are induced by the restriction maps $G_{[t]} \leftarrow G_{[t,t+1]} \to G_{[t+1]}$. The restriction maps $G^F([t],c) \leftarrow G^F([t,t+1],c) \to G^F([t+1],c)$ are given by the canonical projections from the limit of the preceding diagram to $G_{[t]}^{F[t]}(c)$ and $G_{[t+1]}^{F[t+1]}(c)$.

Example 2.14. When $\mathcal{C} := \mathbf{1}$, the terminal category, we are working in the category $[\mathbf{Z}^{\mathsf{op}},\mathbf{Set}]$ of discrete $\mathbf{Set}$-narratives. Exponentials in $\mathbf{Set}$ are precisely hom-sets in $\mathbf{Set}$, so for $F,G\colon \mathbf{Z}^{\mathsf{op}} \to \mathbf{Set}$ we have

$$G^F[t] \cong \mathbf{Set}(F[t],G[t]),$$

while $G^F[t,t+1]$ is the limit of the diagram

$$\mathbf{Set}(F[t],G[t]) \rightarrow \mathbf{Set}(F[t,t+1],G[t]) \leftarrow \mathbf{Set}(F[t,t+1],G[t,t+1]) \rightarrow \mathbf{Set}(F[t,t+1],G[t+1]) \leftarrow \mathbf{Set}(F[t+1],G[t+1]),$$

where the outer maps are given by precomposition with the restriction maps of F, while the inner maps are given by postcomposition with the restriction maps of G. ♢

2.2.2 Subobject Classifier

Every topos $\mathbf{D}$ has a subobject classifier representing the functor $\mathsf{Sub}_{\mathbf{D}}\colon \mathbf{D}^{\mathsf{op}} \to \mathbf{Set}$ that sends each object to the set of (equivalence classes of) its subobjects. If ω is the subobject classifier of $[\mathcal{C},\mathbf{Set}]$, the subobject classifier Ω of the category of discrete $[\mathcal{C},\mathbf{Set}]$-narratives is given like so.

Proposition 2.15. *Suppose ω is the subobject classifier of $[\mathcal{C},\mathbf{Set}]$. For $t \in \mathbb{Z}$ and $c \in \mathcal{C}$, the subobject classifier $\Omega\colon \mathbf{Z}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$ is given as follows:*

$$([t],c) \mapsto \omega(c) \quad \textit{and} \quad ([t,t+1],c) \mapsto \mathbf{Poset}(\mathcal{S},\omega(c)),$$

[6] Note that $G_J^{F_I}(c)$ is *not* necessarily isomorphic to the hom-set $G_I(c)^{F_J(c)}$.

where **Poset** *is the category of posets,* $\mathcal{S}$ *is the walking span category* $[0] \leftarrow [0,1] \rightarrow [1]$*, and* $\omega(c)$ *is endowed with its poset structure given by the subobject relation.*[7]

The restriction maps $\Omega([t],c) \leftarrow \Omega([t,t+1],c) \rightarrow \Omega([t+1],c)$ *are given by precomposition with the inclusions* $\{[0]\} \hookrightarrow \mathcal{S}$ *and* $\{[1]\} \hookrightarrow \mathcal{S}$*.*

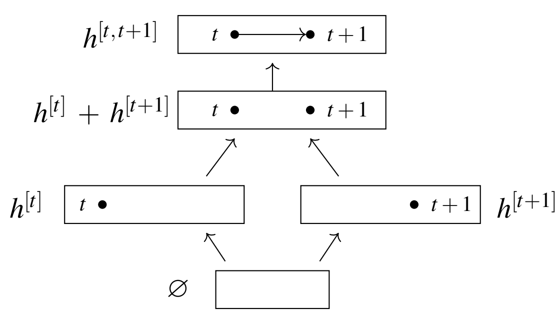
Example 2.16. When $\mathcal{C} := \mathbf{1}$, the subobject classifier Ω of $[\mathbf{Z}^{\mathsf{op}}, \mathbf{Set}]$ can be deduced from the fact that the subobject classifier of **Set** is $\{\varnothing, *\}$ with order $\varnothing \rightarrow *$ (as $\varnothing \subseteq *$). So for $t \in \mathbb{Z}$ we have $\Omega[t] \cong \{\varnothing, *\}$, also with order $\varnothing \rightarrow *$, and (ordered by the subobject relation)

$$\Omega[t,t+1] \cong \{\varnothing \leftarrow \varnothing \rightarrow \varnothing, \quad * \leftarrow \varnothing \rightarrow \varnothing, \quad \varnothing \leftarrow \varnothing \rightarrow *, \quad * \leftarrow \varnothing \rightarrow *, \quad * \leftarrow * \rightarrow *\}$$

with top element $* \leftarrow * \rightarrow *$. The restriction map $\Omega[t,t+1] \rightarrow \Omega[t]$ (resp. $\Omega[t,t+1] \rightarrow \Omega[t+1]$) is given by taking the left (resp. right) leg. More concretely, $\Omega[t] \cong \mathsf{Sub}(h^{[t]}) = \{\varnothing, h^{[t]}\}$, which is isomorphic to $\{\varnothing, *\}$ via $\varnothing \mapsto \varnothing$ and $h^{[t]} \mapsto *$, and

$$\Omega[t,t+1] \cong \mathsf{Sub}(h^{[t,t+1]}) = \{\varnothing, h^{[t]}, h^{[t+1]}, h^{[t]} + h^{[t+1]}, h^{[t,t+1]}\},$$

with restriction maps $\Omega[t,t+1] \rightarrow \Omega[t']$ given by intersection with $h^{[t']}$ for $t' \in \{t, t+1\}$. (Compare this with the subobject classifier for graphs.) We give the poset structure on $\Omega[t,t+1]$ below, drawing its elements as labeled graphs.



We can also derive this poset structure and compute its (bi-)Heyting algebra structure using Catlab.jl; see Listing 2 and Fig. 4. ◊

As is standard for a sheaf topos, we may think of the sections of Ω as local truth values. The elements of $\Omega[t]$ may be interpreted as follows: $\varnothing$ means "false at time t" and $h^{[t]}$ means "true at time t." The elements of $\Omega[t,t+1]$ may be interpreted as follows: $\varnothing$ means "false at times t and $t+1$," $h^{[t]}$ means "true at time t but false at time $t+1$," $h^{[t+1]}$ means "false at time t but true at time $t+1$," $h^{[t]} + h^{[t+1]}$ means "true at times t and $t+1$ but false on $[t,t+1]$," and $h^{[t,t+1]}$ means "true on $[t,t+1]$."

As the subobject classifier of a presheaf category, Ω has an internal (bi-)Heyting algebra structure, so the subobject lattice of each presheaf is a model of (bi-)intuitionistic propositional calculus. So we can perform logical conjunction, disjunction, and implication on the subobjects of a temporal sheaf, providing a structured way to reason about temporal data; all this is implemented in Catlab.jl.

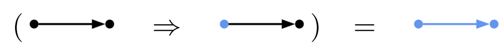
[7] As $\mathcal{S}$ is a posetal category, we may view it as a poset, and it makes sense to discuss order-preserving maps $\mathcal{S} \rightarrow \omega(c)$. Equivalently, we may view the poset $\omega(c)$ as a posetal category and consider functors $\mathcal{S} \rightarrow \omega(c)$. So we could alternatively write $\mathbf{Poset}(\mathcal{S}, \omega(c))$ as $[\mathcal{S}, \omega(c)]$.

```
Ω, subobjects = subobject_classifier(Zop2Inst)

# ... Other code

@withmodel ACSetCategory(Zop2Inst()) (⇒) begin
    S₂ ⇒ S₁ # Implication between two subobjects
end
```

Listing 2: In Catlab.jl, we define two subobjects of a presheaf in $[\mathbf{Z}^{\mathsf{op}}, \mathbf{Set}]$ and perform the implication operation $\Rightarrow$ between them.



$$(\varnothing \quad \Rightarrow \quad h^{[t]}) \quad = \quad h^{[t,t+1]}$$

Figure 4: Graphs depicting two subobjects, $\varnothing$ and $h^{[t]}$, of $h^{[t,t+1]}$; and their implication, $h^{[t,t+1]}$ viewed as a subobject of itself. These graphs are generated from Catlab.jl.

2.3 Temporal Graphs, Cliques, & Paths

In this section, we consider the topos of discrete $[\mathcal{C}, \mathbf{Set}]$-narratives for the walking arrow category $\mathcal{C} := \mathcal{G}$, making $[\mathcal{G}, \mathbf{Set}] = \mathbf{Grph}$ the category of (directed) graphs. We describe a discrete **Grph**-narrative as a time-varying graph, consisting of a discrete **Set**-narrative of vertices and a discrete **Set**-narrative of edges equipped with compatible source and target maps. Equivalently, a persistent (resp. cumulative) discrete **Grph**-narrative is a functor $\mathbf{Z}^{\mathsf{op}} \times \mathcal{G} \to \mathbf{Set}$ (resp. $\mathbf{Z} \times \mathcal{G} \to \mathbf{Set}$).

As in [1, Section 2.5], we would like to port concepts from standard graph theory to the temporal setting, that is to $[\mathbf{Z}^{\mathsf{op}} \times \mathcal{G}, \mathbf{Set}]$ and/or $[\mathbf{Z} \times \mathcal{G}, \mathbf{Set}]$. One way to do this is by phrasing properties of graphs using properties of the topos **Grph**. For instance, consider the concept of a *complete (directed) graph*, in which there is a unique edge between every pair of vertices.

Definition 2.17. A *complete graph* $K(V)$ on a set of vertices V is a (directed) graph whose edge-set is V^2 and whose source and target functions are the first and second projections $V^2 \rightrightarrows V$. We let **KGrph** be the full subcategory of **Grph** spanned by all complete graphs. ◊

The category **KGrph** has limits and colimits given as follows.

Proposition 2.18. *The subcategory inclusion* **KGrph** $\hookrightarrow$ **Grph** *has a left adjoint* L: **Grph** $\hookrightarrow$ **KGrph**, *making* **KGrph** *a reflective subcategory of* **Grph**. *Consequently,* **KGrph** *has all limits, coinciding with limits in* **Grph**, *and all colimits, given by applying* L *to the corresponding colimit in* **Grph**.

In particular, note that in **Grph**, the colimit of complete graphs is not necessarily itself a complete graph and would not necessarily coincide with the same colimit in **KGrph**.

We have the following alternate characterization of complete graphs in terms of representable functors and exponentials in **Grph**.

Proposition 2.19. *Let* $\mathcal{G}(\mathsf{V}, -) \colon \mathcal{G} \to \mathbf{Set}$ *be the graph represented by* $\mathsf{V} \in \mathcal{G}$. *For every graph* $G \in \mathbf{Grph}$, *there is map* $\mathcal{G}(\mathsf{V}, -) \times G \to G$ *that is bijective on vertices that induces a map* $G \to G^{\mathcal{G}(\mathsf{V},-)}$, *where the codomain is the cartesian closure of* $\mathcal{G}(\mathsf{V}, -)$. *We have that* $G^{\mathcal{G}(\mathsf{V},-)} \cong K(G(\mathsf{V}))$, *the complete graph on the vertex-set of* G*; and* G *is a complete graph if and only if the induced map* $G \to G^{\mathcal{G}(\mathsf{V},-)}$ *is an isomorphism of graphs.*

So we propose the following definition for temporally-complete discrete **Grph**-narratives.

Definition 2.20. Given an interval $I \in \mathbf{I}_{\mathbb{Z}}$, let $h^I_{\mathsf{V}} \in [\mathbf{Z}^{\mathsf{op}}, \mathbf{Grph}] \simeq \mathbf{Pe}_{\mathbb{Z}}(\mathbf{Grph})$ be the presheaf equivalent to the sheaf of graphs on $(\mathbf{I}_{\mathbb{Z}}, \mathcal{J}_{\mathbb{Z}})$ given by representable functor $(\mathbf{I}^{\mathsf{op}}_{\mathbb{Z}} \times \mathcal{G})((I, \mathsf{V}), -) = \mathbf{I}_{\mathbb{Z}}(-, I) \times \mathcal{G}(\mathsf{V}, -) \colon \mathbf{I}^{\mathsf{op}}_{\mathbb{Z}} \times \mathcal{G} \to \mathbf{Set}$. We say a persistent discrete **Grph**-narrative $G \in [\mathbf{Z}^{\mathsf{op}}, \mathbf{Grph}]$ is I*-complete* if the induced map $G \to G^{h^I_{\mathsf{V}}}$, where the codomain is the cartesian closure of G and h^I_{V} in $[\mathbf{Z}^{\mathsf{op}}, \mathbf{Grph}]$, is an isomorphism. ◊

We would also like to port the notion of paths from graphs to temporal graphs. Again, our approach is inspired by [1], but we clarify a few points. We define first *path graphs* that represent (directed) *paths*.

Definition 2.21. For $n \in \mathbb{N}$,[8] the *length-n path graph* $P_n \in \mathbf{Grph}$ is the (directed) graph with vertex-set $\{0,\dots,n\}$, edge-set $\{1,\dots,n\}$, source function $k \mapsto k-1$, and target function $k \mapsto k$:

$$0 \xrightarrow{1} 1 \xrightarrow{2} \cdots \xrightarrow{n} n.$$

A *length-n path* of a graph $G \in \mathbf{Grph}$ is a map $p\colon P_n \to G$; the path is *simple* if p is monomorphic. ♢

We want to build sheaves or cosheaves of graphs consisting only of path graphs, but not all limits and colimits of path graphs in **Grph** are path graphs themselves. There is a special kind of pushout of path graphs that does, however, always yield another path graph. These pushouts are built from the following inclusions of path graphs.

Definition 2.22. For $m,n \in \mathbb{N}$ with $m \leq n$, call the map $p_n^m\colon P_m \to P_n$ sending vertex i for $0 \leq i \leq m$ in P_m to vertex i in P_n a *prefix map* and the map $s_n^m\colon P_m \to P_n$ sending vertex i in P_m to vertex $i-m+n$ in P_n a *suffix map*. ♢

Lemma 2.23. *Given $\ell,m,n \in \mathbb{N}$, we have the following pushout square in* **Grph**:

$$\begin{array}{ccccc} & & P_{m+n-\ell} & & \\ & \overset{p^m_{m+n-\ell}}{\nearrow} & \lrcorner & \overset{s^n_{m+n-\ell}}{\nwarrow} & \\ P_m & & & & P_n \\ & \underset{s^\ell_m}{\nwarrow} & & \underset{p^\ell_n}{\nearrow} & \\ & & P_\ell & & \end{array}$$

Corollary 2.24. *Let $P\colon \mathbf{Z} \to \mathbf{Grph}$ be a functor satisfying the following:*
- *for all $I \in \mathbf{Z}$, the graph $P(I)$ is a path graph;*
- *for all $t \in \mathbb{Z}$, the functor P sends the map $[t] \to [t,t+1]$ in $\mathbf{Z}$ to a prefix map; and*
- *for all $t \in \mathbb{Z}$, the functor P sends the map $[t] \to [t-1,t]$ in $\mathbf{Z}$ to a suffix map.*

Then the cosheaf in $\mathbf{Cu}_{\mathbb{Z}}(\mathbf{Grph})$ corresponding to P under the equivalence $\mathbf{Cu}_{\mathbb{Z}}(\mathbf{Grph}) \simeq [\mathbf{Z},\mathbf{Grph}]$ from Proposition 2.3 sends every $I \in \mathbf{I}_{\mathbb{Z}}$ to (a graph isomorphic to) a path graph.

Definition 2.25. Let $P\colon \mathbf{Z} \to \mathbf{Grph}$ be a functor satisfying the three conditions given in Corollary 2.24. A *temporal path* of a cumulative discrete **Grph**-narrative $G\colon \mathbf{Z} \to \mathbf{Grph}$ is a natural transformation $p\colon P \to G$; the temporal path is *simple* if p is monomorphic. ♢

We give an example of the use of temporal paths in public health in Section 3.1.

3 Public Health Applications

Effective control of communicable disease outbreaks demands rapid public health response supported by mandated, standardized reporting. Here we present two contrasting public health applications to underscore the diversity of applications within the public health sphere.

[8] Our set of natural numbers $\mathbb{N}$ contains 0.

3.1 Temporal Paths and Pathogen Spread

Contact tracing serves as a key process to infectious disease control across a broad array of transmissible infections. Contact tracing delineates "cases"—individuals confirmed to be infected—and their "contacts" who commonly remain uninfected. When a case mentions having exposed to a contact over a specified time interval, contact tracers consider them connected in an undirected temporal "egocentric" network centered on that case. Often, networks for different cases will be glued together on contacts that they share. Because each edge is specific to a particular temporal interval, any two individuals within the network can have a natural number of edges between them, each for disjoint time intervals.

Understanding of the process of infection spread within such networks commonly benefits from reconstructing who might have infected whom. Such reconstruction is important for contact tracing: if no individuals in a reconstructed case-contact network are capable of having transmitted to a given case, further tracing may be needed. To reason about possible routes of infection, the network elicited from the contact tracing is scrutinized to determine whether a case diagnosed earlier (A) could have transmitted to one diagnosed later (C). As each edge within a network is associated with a specific time interval over which case interaction transpired, the challenge here lies in finding how a pathogen may have transmitted incrementally from A to C over a period of time despite there not being a contiguous path extending from A to C within at any one time.

For such transmission, the pathogen needs to cross successive edges from A to C associated with a non-decreasing temporal sequence. If such edges from A to C are out of temporal order, infection will not follow this path to spread from A to C. Hence, to determine if infection can be transmitted from A to C, one needs consider a *temporal path* from A to C.

Definition 2.25 of a temporal path within a temporal graph as a natural transformation between **Grph**-narratives allows for assessing whether an infection starting at one person could transmit to another person along successive network edges within a compatible time sequence. Given individuals A and C, we use hom-finding in the category of temporal narratives to derive whether there exists a temporal path from the vertex corresponding to A to that corresponding to C. To do this, we consider a cumulative discrete **Grph**-narrative: $G\colon \mathbf{Z} \to \mathbf{Grph}$. It bears emphasis that each $G[t]$ is a copresheaf $\mathcal{G} \to \mathbf{Set}$ on the graph schema, representing the case-contact graph within a given time interval. Likewise, $G[t,t+1]$ is a graph, one expressing the cumulative behaviour over that interval.

We now utilize Definition 2.25 and consider finding a temporal path in G by defining a natural transformation from the temporal path graph P in the sense of Corollary 2.24 to the discrete **Grph**-narrative G. As a natural transformation between graph narratives, it has a component graph morphism for each time point or interval in $\mathbf{Z}$. Although the apex $(G[t,t+1])$ characterizes an aggregate (summary) graph, the naturality condition enforces the temporal ordering such that the path found has the edges sequenced in an order that infection can successively "hop" from A to C. Figure 5 illustrates the naturality squares involved, which enforce proper sequencing.

The presence of a temporal path from A to C in a temporal graph narrative thus indicates whether A could have infected C. To identify whether person A could have identified person C, we can search for temporal paths of successively longer edge counts k. For a given k, we consider simple temporal paths $P \hookrightarrow G$, where temporal path graph P satisfying the criteria of Corollary 2.24, and in which, for all $I \in \mathbf{Z}$, the graph $P(I)$ is given by a length-k path graph P_k.

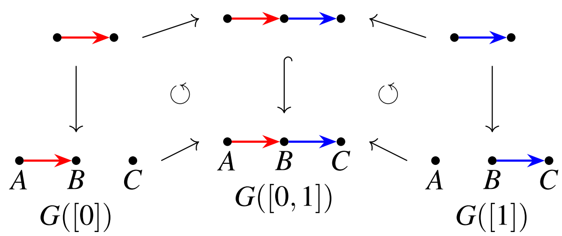


Figure 5: Finding temporal paths in temporal graphs ensures proper edge sequencing.

3.2 Prevalence across an Agent-Based Population over Time

Critical to public health ABMs and empirical datasets is reasoning about those affected by a given condition over differing time intervals. Two aspects to this investigation is identifying those who had a condition *at any point within* a time interval or a condition *throughout* a time interval. We consider the case where each time, $[t]$, is associated with a set of people having some condition. We simplify our focus to conditions that, once incurred, do not disappear (e.g., Diabetes mellitus, dementia, and pathogens such as herpes and chickenpox). We consider persistent and cumulative perspectives for such narratives over both unit and broader intervals.

A persistent perspective on such data would characterize for each interval, individuals who carry that condition throughout that time interval. Consider persistent narrative $F: \mathbf{Z}^{op} \to \mathbf{Set}$. Data $F[t,t+1]$ for unit intervals represents the apex of spans which relate individuals $F[t]$ at to $F[t+1]$. As with other persistent perspectives, this would characterize the value for interval $F[t_a,t_c]$ of more than unit length as the pullback of $F[t_a,t_b] \times_{F[t_b]} F[t_b,t_c]$—a quantity that the sheaf condition ensures is invariant up to isomorphism for any pair of intervals $[t_a,t_b]$ and $[t_b,t_c]$. This pullback is fibered over people shared across such intervals—those at time $[t_b]$—thus ensuring that the only individuals counted will be those who are present in both intervals.

By contrast, a cumulative perspective on such data would allow, for unit interval $[t,t+1]$, a total view of all the distinct individuals with that conditions between time t and $t+1$. Consider cumulative narrative $G\colon \mathbf{Z} \to \mathbf{Set}$. $G[t,t+1]$ represents the apex of a cospan of individuals with the condition at any time within that interval, and which has inclusions from those $G[t]$ with the condition at time t and those $G[t+1]$ at time $t+1$. For longer intervals, such cumulative totals would result from pushouts $F[t_a,t_b] +_{F[t_b]} F[t_b,t_c]$, where the total across two time intervals would be fibered over $F[t_b]$, the people held in common as having the condition in those two intervals. As above, the guarantee of Johnstone coverage ensures that this value is invariant, up to isomorphism regardless of the choice of t_b.

4 Conclusion

Future work will expand on the topos theory applied here and build on the given public health applications. Such vignettes motivate further translational efforts for practical use in ABMs or to empirical data. While Catlab.jl can compute $[\mathbf{Z}^{\mathbf{op}}, \mathbf{Set}]$ and explore the logic of its subobjects, we have yet to express logical quantifiers in the internal logic of the topos of temporal sheaves in code. We also hope to leverage the geometric morphisms given in Proposition 2.8 to implement structured comparisons of data across different timescales, including situations when not all data is known. Finally, building upon this paper's explanatory nature, we plan to develop pedagogical material around temporal sheaf theory further accessible to broader audiences.

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A Appendix: Proofs of results

We give proofs for assertions throughout the text. Our first result is used implicitly in Definition 2.1.

Proposition A.1. *There is a coverage on* $\mathbf{I}_\tau$ *given by the families* $\{[a,b]\}$ *and* $\{[a,p],[p,b]\}$ *covering* $[a,b] \in \mathbf{I}_\tau$ *for all* $p \in \tau$ *satisfying* $a \leq p \leq b$.

Proof. As $\mathbf{I}_\tau$ is a poset, we follow the definition of a coverage on a poset given in [6]. Given $[a',b'] \in \mathbf{I}_\tau$ with $[a',b'] \subseteq [a,b]$, we have two cases: either $a' \leq p \leq b'$ or not. In the former case, each interval in the cover $\{[a',p],[p,b']\}$ of $[a',b']$ is contained in an interval in the cover $\{[a,p],[p,b]\}$ of $[a,b]$: we have $[a',p] \subseteq [a,p]$ and $[p,b'] \subseteq [p,b]$. In the latter case, either $a \leq a' \leq b' < p \leq b$ or $a \leq p < a' \leq b' \leq b$, so the lone interval in the cover $\{[a',b']\}$ of $[a',b']$ is contained in one of the intervals in $\{[a,p],[p,b]\}$. $\square$

Proof of Proposition 2.3. We prove the result for persistent narratives; the result for cumulative narratives follows dually. First we show that the functor $-\big|_{\mathbf{Z}}\colon \mathbf{Pe}_{\mathbb{Z}}(\mathbf{D}) \to [\mathbf{Z}^{\mathsf{op}}, \mathbf{D}]$ given by restriction along the inclusion $\mathbf{Z}^{\mathsf{op}} \hookrightarrow \mathbf{I}_{\mathbb{Z}}^{\mathsf{op}}$ is essentially surjective. Given $X\colon \mathbf{Z}^{\mathsf{op}} \to \mathbf{D}$, extend it to a sheaf $\overline{X}\colon \mathbf{I}_{\mathbb{Z}}^{\mathsf{op}} \to \mathbf{D}$ on $(\mathbf{I}_{\mathbb{Z}}, \mathcal{J}_{\mathbb{Z}})$ that restricts to X by inductively constructing, for every $n \geq 2$, each set $\overline{X}[t,t+n]$ with restriction maps to $\overline{X}[t,t+n-1]$ and $\overline{X}[t+1,t+n]$ for $t \in \mathbb{Z}$ via the following pullback:

$$
\begin{array}{ccccc}
 & & \overline{X}[t,t+n] & & \\
 & \swarrow & \lrcorner & \searrow & \\
\overline{X}[t,t+n-1] & & & & \overline{X}[t+1,t+n] \\
 & \searrow & & \swarrow & \\
 & & \overline{X}[t+1,t+n-1]. & &
\end{array}
$$

The remaining restriction maps of $\overline{X}$ are given by composing these restriction maps along the covering relations $[t,t+n-1],[t+1,t+n] \subseteq [t,t+n]$, with the commutativity of each square above ensuring well-definedness and functoriality; and the sheaf condition is satisfied by the pasting law for pullbacks.

To show that $-\big|_{\mathbf{Z}}$ is fully faithful, it suffices to show that for $X,Y\colon \mathbf{Z}^{\mathsf{op}} \to \mathbf{D}$ and natural transformation $\alpha\colon X \to Y$, there is a unique natural transformation $\overline{\alpha}\colon \overline{X} \to \overline{Y}$ that restricts to α on $\mathbf{Z}^{\mathsf{op}}$. But this follows inductively from the universal property of pullbacks. $\square$

Proof of Proposition 2.4. The isomorphism as given is evidently bijective on objects. It is also appropriately order reversing, sending $[t] \subseteq [t,t+1]$ to $[t,t+1] \supseteq [t+1]$ and $[t+1] \subseteq [t,t+1]$ to $[t+1,t+2] \supseteq [t+1]$. $\square$

Proof of Proposition 2.6. The category of elements of $\mathcal{P}_{\mathbb{Z}}\colon \mathcal{G} \to \mathbf{Set}$ consists of:

- objects (E,t) and (V,t) for each $t \in \mathbb{Z}$;
- morphisms $(\mathsf{E},t) \to (\mathsf{V},t)$ and $(\mathsf{E},t) \to (\mathsf{V},t+1)$ for each $t \in \mathbb{Z}$.

There is therefore an isomorphism from $\mathbf{Z}^{\mathsf{op}}$ to this category sending $[t] \mapsto (\mathsf{V},t)$ and $[t,t+1] \mapsto (\mathsf{E},t)$ on objects as well as $[t] \to [t,t+1]$ to $(\mathsf{E},t) \to (\mathsf{V},t)$ and $[t+1] \to [t,t+1]$ to $(\mathsf{E},t) \to (\mathsf{V},t+1)$ on morphisms for $t \in \mathbb{Z}$. $\square$

Proof of Proposition 2.8. First, we will show that a monotone map $r\colon \tau \to \tau'$ induces a monotone map $\mathbf{I}_\tau \to \mathbf{I}_{\tau'}$ that we also call r, sending $[a,b] \mapsto [r(a), r(b)]$. Order is preserved: if $[a_1,b_1] \subseteq [a_2,b_2]$ in $\mathbf{I}_\tau$, equivalent to $a_2 \le a_1 \le b_1 \le b_2$ in τ, then $[r(a_1), r(b_1)] \subseteq [r(a_2), r(b_2)]$ in $\mathbf{I}_\tau$, equivalent to $r(a_2) \le r(a_1) \le r(b_1) \le r(b_2)$ in τ'.

Next we show that $r\colon \mathbf{I}_\tau \to \mathbf{I}_{\tau'}$ is (representably) flat as a functor (defined in [4] and stated for the case of posets in [6]) whenever r satisfies the given condition. Indeed, the condition ensures that for every $[c'_1, c'_2] \in \tau'$, there exist $a, b \in \tau$ with $r(a) \le c'_1 \le c'_2 \le r(b)$ and thus $[c'_1, c'_2] \subseteq [r(a), r(b)]$. Then for $[a_1,b_1],[a_2,b_2] \in \mathbf{I}_\tau$, assume without loss of generality $a_1 \le a_2$ and, if $a_1 = a_2$, then $b_2 \le b_1$. We have three cases: either $a_1 \le b_1 < a_2 \le b_2$, in which case $[a_1,b_1]$ and $[a_2,b_2]$ share no lower bound, and neither do $[r(a_1), r(b_1)]$ and $[r(a_2), r(b_2)]$; or we have $a_1 \le a_2 \le b_2 \le b_1$, in which case $[a_1,b_1] \cap [a_2,b_2] = [a_2,b_2]$ and $[r(a_1), r(b_1)] \cap [r(a_2), r(b_2)] = [r(a_2), r(b_2)]$; or $a_1 < a_2 \le b_1 < b_2$, so $[a_1,b_1] \cap [a_2,b_2] = [a_2,b_1]$ and $[r(a_1), r(b_1)] \cap [r(a_2), r(b_2)] = [r(a_2), r(b_1)]$.

It remains to show that r preserves covering families: for all $a \le p \le b$ in τ, we have that $\{[r(a), r(b)]\}$ and $\{[r(a), r(p)], [r(p), r(b)]\}$ both cover $r([a,b])$ in $(\mathbf{I}_{\tau'}, \mathcal{J}_{\tau'})$. Therefore $r\colon (\mathcal{I}_\tau, \mathcal{J}_\tau) \to (\mathcal{I}_{\tau'}, \mathcal{J}_{\tau'})$ is a morphism of sites (as in [5]), so it induces a geometric morphism of topoi.

Conversely, assume r does not satisfy the given condition. Say there exists $c' \in \tau'$ such that either $c' > r(c)$ for all $c \in \tau$ or $c' < r(c)$ for all $c \in \tau$. In either case, no $[a,b] \in \mathbf{I}_\tau$ satisfies $[c',c'] \subseteq r([a,b])$, so $\mathbf{I}_{\tau'}([c',c'], r([a,b])) \cong \varnothing$. Composition with $r\colon \mathbf{I}_\tau \to \mathbf{I}_{\tau'}$ yields a functor $\mathbf{Pe}_{\tau'}(\mathbf{Set}) \to \mathbf{Pe}_\tau(\mathbf{Set})$ whose left adjoint is necessarily the composite

$$\mathbf{Pe}_\tau(\mathbf{Set}) \hookrightarrow [\mathbf{I}_\tau^{\mathsf{op}}, \mathbf{Set}] \xrightarrow{\mathrm{Lan}_r} [\mathbf{I}_{\tau'}^{\mathsf{op}}, \mathbf{Set}] \xrightarrow{L_{\mathcal{J}_{\tau'}}} \mathbf{Pe}_{\tau'}(\mathbf{Set}),$$

where Lan_r is the left Kan extension along r and $L_{\mathcal{J}_{\tau'}}$ is sheafification. For this to be the left adjoint in a geometric morphism, it must preserve finite limits, particularly the constant presheaf $*$ to the terminal set that is also the terminal sheaf. So Lan_r needs to preserve $*$. But by the coend formula for left Kan extensions of presheaves,

$$(\mathrm{Lan}_r *)[c',c'] \cong \int^{[a,b]\in\mathbf{I}_\tau} \mathbf{I}_{\tau'}([c',c'], r([a,b])) \times *([a,b]) \cong \int^{[a,b]\in\mathbf{I}_\tau} \varnothing \times * \cong \varnothing \not\cong *.$$

So r does not induce a geometric morphism. □

Proof of Proposition 2.10. Fix $[c,d] \in \mathbf{I}_\tau$. The representable $h^{[c,d]}\colon \mathbf{I}_\tau^{\mathsf{op}} \to \mathbf{Set}$ sends $[c',d'] \in \mathbf{I}_\tau$ to $*$ if $[c',d'] \subseteq [c,d]$ and $\varnothing$ otherwise. Then for $a, p, b \in \tau$ with $a \le p \le b$, every set in the square

$$\begin{array}{ccc} & h^{[c,d]}[a,b] & \\ \swarrow & & \searrow \\ h^{[c,d]}[a,p] & & h^{[c,d]}[p,b] \\ \searrow & & \swarrow \\ & h^{[c,d]}[p,p] & \end{array}$$

is $\varnothing$ or $*$, and by case-analysis its commutativity ensures it is a pullback unless the top set is $\varnothing$ and the bottom three sets are $*$. But if $h^{[c,d]}[a,p] \cong h^{[c,d]}[p,b] \cong *$, then $[a,p],[p,b] \subseteq [c,d]$, so $[a,b] \subseteq [c,d]$ as well, making $h^{[c,d]}[a,b] \cong *$. Hence $h^{[c,d]}$ is a sheaf in $\mathbf{Pe}_\tau(\mathbf{Set})$. □

Proof of Proposition 2.13. For $I \in \mathbf{Z}$, by the Yoneda Lemma and cartesian closure,

$$G^F(I,c) \cong [\mathbf{Z}^{\mathsf{op}} \times \mathcal{C}, \mathbf{Set}](\mathbf{Z}(-,I) \times \mathcal{C}(c,=) \times F, G).$$

When $I \coloneqq [t]$, the representable functor $h^{[t]} = \mathbf{Z}(-,[t])$ sends all intervals to $\varnothing$ except for $[t]$ itself, which it sends to $*$. It follows that a natural transformation $h^{[t]}(-) \times \mathcal{C}(c,=) \times F \to G$ is completely determined by its $([t],c')$-components for $c' \in \mathcal{C}$, which comprise a natural transformation $\mathcal{C}(c,-) \times F_{[t]} \to G_{[t]}$ between functors $\mathcal{C} \to \mathbf{Set}$. Then leveraging cartesian closure and the Yoneda lemma on $[\mathcal{C},\mathbf{Set}]$,

$$\begin{aligned} G^F([t],c) &\cong [\mathcal{C},\mathbf{Set}](\mathcal{C}(c,-)\times F_{[t]}, G[t]) \\ &\cong [\mathcal{C},\mathbf{Set}](\mathcal{C}(c,-), G_{[t]}^{F_{[t]}}) \\ &\cong G_{[t]}^{F_{[t]}}(c) \end{aligned}$$

natural in c.

If $I \coloneqq [t,t+1]$, then $h^{[t,t+1]} = \mathbf{Z}(-,[t,t+1])$ sends all intervals to $\varnothing$ except for $[t]$, $[t+1]$, and $[t,t+1]$, which it sends to $*$. So a natural transformation $h^{[t,t+1]}(-) \times \mathcal{C}(c,=) \times F \to G$ is determined by its (J,c')-components for $c' \in \mathcal{C}$ for $J \in \{[t],[t+1],[t,t+1]\}$, giving rise to a natural transformation $\alpha_J \colon \mathcal{C}(c,-) \times F_J \to G_J$ between functors $\mathcal{C} \to \mathbf{Set}$ for each $J \in \{[t],[t+1],[t,t+1]\}$, such that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathcal{C}(c,-)\times F_{[t,t+1]} & & \\ & \swarrow & & \searrow & \\ \mathcal{C}(c,-)\times F_{[t]} & & \downarrow{\scriptstyle \alpha_{[t,t+1]}} & & \mathcal{C}(c,-)\times F_{[t+1]} \\ \downarrow{\scriptstyle \alpha_{[t]}} & & G_{[t,t+1]} & & \downarrow{\scriptstyle \alpha_{[t+1]}} \\ & \swarrow & & \searrow & \\ G_{[t]} & & & & G_{[t+1]}. \end{array}$$

By the cartesian closure of $[\mathcal{C},\mathbf{Set}]$, every natural transformation $\mathcal{C}(c,-) \times F_J \to G_K$ for $J,K \in \mathbf{Z}$ is equivalently a natural transformation $\mathcal{C}(c,-) \to G_K^{F_J}$, which is in turn equivalently an element of $G_K^{F_J}(c)$ by the Yoneda Lemma. So we may equivalently write the triple of natural transformations $(\alpha_{[t]},\alpha_{[t,t+1]},\alpha_{[t+1]})$ as a triple $(a_{[t]},a_{[t,t+1]},a_{[t+1]})$ with each $a_J \in G_J^{F_J}(c)$ such that the following diagram commutes:[9]

$$\begin{array}{ccccc} & & * & & \\ & \overset{a_{[t]}}{\swarrow} & & \overset{a_{[t+1]}}{\searrow} & \\ G_{[t]}^{F_{[t]}}(c) & & \downarrow{\scriptstyle a_{[t,t+1]}} & & G_{[t+1]}^{F_{[t+1]}}(c) \\ \downarrow & & G_{[t,t+1]}^{F_{[t,t+1]}}(c) & & \downarrow \\ & \swarrow & & \searrow & \\ G_{[t]}^{F_{[t,t+1]}}(c) & & & & G_{[t+1]}^{F_{[t,t+1]}}(c). \end{array}$$

Hence $G^F([t,t+1],c)$ is the limit of the lower W-shaped part of the diagram, and the result follows. □

We also give the dual result for cumulative $[\mathcal{C},\mathbf{Set}]$-narratives $F,G\colon \mathbf{Z}\times\mathcal{C} \to \mathbf{Set}$. Again, for $I,J \in \mathbf{Z}$, we write $F_I(c) \coloneqq F(I,c)$ and $G_J(c) \coloneqq G(J,c)$, so that $F_I, G_J \in [\mathcal{C},\mathbf{Set}]$ and $G_J^{F_I}$ is their cartesian closure.

[9] Here we identify an element of a set $G_J^{F_J}(c)$ with a function $* \to G_J^{F_J}(c)$ that picks out that element.

Proposition A.2. *For $t \in \mathbb{Z}$ and $c \in \mathcal{C}$, the cartesian closure $G^F : \mathbf{Z} \times \mathcal{C} \to \mathbf{Set}$ sends*

$$([t,t+1],c) \mapsto G^{F_{[t,t+1]}}_{[t,t+1]}(c)$$

and $([t],c)$ to the limit of the diagram

$$\begin{array}{ccccccc} & G^{F_{[t]}}_{[t-1,t]}(c) & & & G^{F_{[t]}}_{[t,t+1]}(c) & & \\ \nearrow & & \nwarrow & & \nearrow & & \nwarrow \\ G^{F_{[t-1,t]}}_{[t-1,t]}(c) & & & G^{F_{[t]}}_{[t]}(c) & & & G^{F_{[t,t+1]}}_{[t,t+1]}(c), \end{array}$$

where the outer maps are induced by the pair of extension maps $F_{[t-1,t]} \leftarrow F_{[t]} \to F_{[t,t+1]}$ while the inner maps are induced by the extension maps $G_{[t-1,t]} \leftarrow G_{[t]} \to G_{[t,t+1]}$.

The extension maps $G^F([t-1,t],c) \leftarrow G^F([t],c) \to G^F([t,t+1],c)$ are given by the canonical projections from the limit of the preceding diagram to $G^{F_{[t-1,t]}}_{[t-1,t]}(c)$ and $G^{F_{[t,t+1]}}_{[t,t+1]}(c)$.

Proof. The result follows from Proposition 2.13 via conjugation with the isomorphism $\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}^{\mathsf{op}}$ from Proposition 2.4 sending $[t] \mapsto [t,t+1]$ and $[t,t+1] \mapsto [t+1]$. □

Proof of Proposition 2.15. By the universal property of the subobject classifier and the Yoneda Lemma,

$$\Omega(I,c) \cong \mathrm{Sub}_{[\mathbf{Z}^{\mathsf{op}} \times \mathcal{C}, \mathbf{Set}]}(h^I(-) \times \mathcal{C}(c,=))$$

for $I \in \mathbf{Z}$ and $c \in \mathcal{C}$, with $h^I = \mathbf{Z}(-,I)$. When $I \coloneqq [t]$ for $t \in \mathbb{Z}$, for $J \in \mathbf{Z}$, we have $h^{[t]}(J) = \varnothing$ whenever $J \neq [t]$, so every subpresheaf of $h^{[t]}(-) \times \mathcal{C}(c,=)$ must send $(J,c') \mapsto \varnothing$ for $c' \in \mathcal{C}$ whenever $J \neq [t]$ as well. Meanwhile $h^{[t]}[t] \cong *$, so $h^{[t]}[t] \times \mathcal{C}(c,-) \cong \mathcal{C}(c,-)$. We can therefore identify the subpresheaves of $h^{[t]}(-) \times \mathcal{C}(c,=)$ in $[\mathbf{Z}^{\mathsf{op}} \times \mathcal{C}, \mathbf{Set}]$ with subpresheaves of $\mathcal{C}(c,-)$ in $[\mathcal{C}, \mathbf{Set}]$:

$$\Omega([t],c) \cong \mathrm{Sub}_{[\mathcal{C},\mathbf{Set}]}(\mathcal{C}(c,-)) \cong \omega(c).$$

Similarly, if $I \coloneqq [t,t+1]$, we have $h^{[t,t+1]}(J) = \varnothing$ whenever $J \not\subseteq [t,t+1]$ and $h^{[t,t+1]}(J) = *$ whenever $J \subseteq [t,t+1]$, so subpresheaves of $h^{[t,t+1]}(-) \times \mathcal{C}(c,=)$ in $[\mathbf{Z}^{\mathsf{op}} \times \mathcal{C}, \mathbf{Set}]$ can be identified with subfunctors of the functor $\mathcal{S} \to [\mathcal{C}, \mathbf{Set}]$ given by the diagram

$$\begin{array}{ccccc} & & \mathcal{C}(c,-) & & \\ & /\!\!/ & & \backslash\!\!\backslash & \\ \mathcal{C}(c,-) & & & & \mathcal{C}(c,-). \end{array}$$

Such a subfunctor $S\colon \mathcal{S} \to [\mathcal{C}, \mathbf{Set}]$ can be added to this commutative diagram like so:

$$\begin{array}{ccccc} & & S[0,1] & & \\ & \swarrow & \downarrow & \searrow & \\ S[0] & & & & S[1] \\ \downarrow & & \mathcal{C}(c,-) & & \downarrow \\ & /\!\!/ & & \backslash\!\!\backslash & \\ \mathcal{C}(c,-) & & & & \mathcal{C}(c,-). \end{array}$$

By the properties of monomorphisms, the upper diagonal maps $S[0] \leftarrow S[0,1] \rightarrow S[1]$ must be monomorphisms as well. As the set of subobjects $\mathbf{Sub}_{[\mathcal{C},\mathbf{Set}]}(\mathcal{C}(c,-))$ of $\mathcal{C}(c,-)$ is precisely $\omega(c)$, the set $\omega(c)$ inherits the poset structure on $\mathbf{Sub}_{[\mathcal{C},\mathbf{Set}]}(\mathcal{C}(c,-))$, and S may be identified with an order-preserving map $\mathcal{S} \rightarrow \omega(c)$: a pair of elements of $\omega(c)$ and a lower bound for both. Hence

$$\Omega([t,t+1],c) \cong \mathbf{Poset}(\mathcal{S}, \omega(c))$$

and the result follows. □

Again, we give the dual result for the subobject classifier of the category of cumulative $[\mathcal{C},\mathbf{Set}]$-narratives $[\mathbf{Z} \times \mathcal{C}, \mathbf{Set}]$. We still use ω to denote the subobject classifier of $[\mathcal{C},\mathbf{Set}]$.

Proposition A.3. *For $t \in \mathbb{Z}$ and $c \in \mathcal{C}$, the subobject classifier $\Omega\colon \mathbf{Z} \times \mathcal{C} \rightarrow \mathbf{Set}$ sends*

$$([t],c) \mapsto \mathbf{Poset}(\mathcal{S}, \omega(c)) \quad \text{and} \quad ([t,t+1],c) \mapsto \omega(c),$$

*where **Poset** is the category of posets, $\mathcal{S}$ is the walking span category $[0] \leftarrow [0,1] \rightarrow [1]$, and $\omega(c)$ is endowed with its poset structure given by the subobject relation.*

The extension maps $\Omega([t-1,t],c) \leftarrow \Omega([t],c) \rightarrow \Omega([t,t+1],c)$ are given by precomposition with the inclusions $\{[0]\} \hookrightarrow \mathcal{S}$ and $\{[1]\} \hookrightarrow \mathcal{S}$.

Proof. The result follows from composing the subobject classifier from Proposition 2.15 with the isomorphism $\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}^{\mathsf{op}}$ from Proposition 2.4 sending $[t] \mapsto [t,t+1]$ and $[t,t+1] \mapsto [t+1]$. □

Proof of Proposition 2.18. For each graph $G \in \mathbf{Grph}$, consider the complete graph $K(G(\mathsf{V}))$ on the vertex-set of G. It suffices to find a universal graph morphism $\eta_G\colon G \rightarrow K(G(\mathsf{V}))$ such that for every complete graph $C \in \mathbf{KGrph}$ and every graph morphism $g\colon G \rightarrow C$, there exists a unique graph morphism $f\colon K(G(\mathsf{V})) \rightarrow C$ with $f \circ \eta_G = g$.

Define $\eta_G\colon G \rightarrow K(G(\mathsf{V}))$ as follows. As G and $K(G(\mathsf{V}))$ have the same vertex-set $G(\mathsf{V})$, we may let η_G be the identity on vertices. On edges, η_G must then send each edge $e\colon v \rightarrow w$ of G to the unique edge $v \rightarrow w$ of $K(G(\mathsf{V}))$.

Then for every complete graph C and graph morphism $g\colon G \rightarrow C$, if there exists a graph morphism $f\colon K(G(\mathsf{V})) \rightarrow C$ satisfying $f \circ \eta_G = g$, then f and g must coincide on vertices. Then on edges, each edge $v \rightarrow w$ of $K(G(\mathsf{V}))$ must be sent by f to the unique edge $g(v) \rightarrow g(w)$ in C. Hence f is uniquely defined, and we can check that it satisfies $f \circ \eta_G = g$ on edges: given an edge $e\colon v \rightarrow w$ of G, the map η_G will send it to the unique edge $v \rightarrow w$ of $K(G(\mathsf{V}))$, which f then sends to the unique edge $g(v) \rightarrow g(w)$ of C, agreeing with where g sends $e\colon v \rightarrow w$. Therefore the inclusion $\mathbf{KGrph} \hookrightarrow \mathbf{Grph}$ is a right adjoint, and **KGrph** is a reflective subcategory of **Grph**.

It follows from standard results on reflective subcategories (see e.g. [7, Proposition 4.5.15(i)]) that the subcategory inclusion creates all limits in **Grph**. As a presheaf category, **Grph** has all (small) limits, so **KGrph** does too, and they coincide with those in **Grph**. Moreover, **KGrph** has all colimits that **Grph** has, formed by applying the left adjoint to the subcategory inclusion to the colimit in **Grph**. As a presheaf category, **Grph** also has all (small) colimits. □

Proof of Proposition 2.19. The graph $\mathcal{G}(\mathsf{V},-)$ has vertex-set $\mathcal{G}(\mathsf{V},\mathsf{V}) = \{\mathsf{id}_\mathsf{V}\}$ as well as edge-set $\mathcal{G}(\mathsf{V},\mathsf{E}) = \varnothing$, so the graph $\mathcal{G}(\mathsf{V},-) \times G$ has vertex-set $\{\mathsf{id}_\mathsf{V}\} \times G(\mathsf{V}) \cong G(\mathsf{V})$, the vertex-set of G, and edge-set $\varnothing \times G(\mathsf{E}) \cong \varnothing$. So $\mathcal{G}(\mathsf{V},-)$ is the graph with 1 vertex and no edges. Then $\mathcal{G}(\mathsf{V},-) \times \mathcal{G}(\mathsf{V},-) \cong \mathcal{G}(\mathsf{V},-)$. Moreover, there is a graph morphism $\mathcal{G}(\mathsf{V},-) \times G \rightarrow G$ that is an isomorphism on vertices and uniquely determined on edges.

By the Yoneda Lemma and the cartesian closure of **Grph**,

$$G^{\mathcal{G}(\mathsf{V},-)}(\mathsf{V}) \cong \mathbf{Grph}(\mathcal{G}(\mathsf{V},-), G^{\mathcal{G}(\mathsf{V},-)}) \cong \mathbf{Grph}(\mathcal{G}(\mathsf{V},-)\times\mathcal{G}(\mathsf{V},-), G) \cong \mathbf{Grph}(\mathcal{G}(\mathsf{V},-), G) \cong G(\mathsf{V}),$$

and the induced map $G^{\mathcal{G}(\mathsf{V},-)} \to G$ is an isomorphism on vertices as well. Meanwhile

$$G^{\mathcal{G}(\mathsf{V},-)}(\mathsf{E}) \cong \mathbf{Grph}(\mathcal{G}(\mathsf{E},-), G^{\mathcal{G}(\mathsf{V},-)}) \cong \mathbf{Grph}(\mathcal{G}(\mathsf{E},-)\times\mathcal{G}(\mathsf{V},-), G).$$

As a graph, $\mathcal{G}(\mathsf{E},-)$ has vertex-set $\mathcal{G}(\mathsf{E},\mathsf{V}) = \{\mathsf{s},\mathsf{t}\}$ and edge-set $\mathcal{G}(\mathsf{E},\mathsf{E}) = \{\mathrm{id}_{\mathsf{E}}\}$, whose sole edge has source vertex s and target vertex t. That is, $\mathcal{G}(\mathsf{E},-)$ consists of 2 vertices and 1 edge between them. Then $\mathcal{G}(\mathsf{E},-)\times\mathcal{G}(\mathsf{V},-)$ has $2\cdot 1 = 2$ vertices and $1\cdot 0 = 0$ edges, so it is isomorphic to the coproduct $\mathcal{G}(\mathsf{V},-)+\mathcal{G}(\mathsf{V},-)$. Hence

$$G^{\mathcal{G}(\mathsf{V},-)}(\mathsf{E}) \cong \mathbf{Grph}(\mathcal{G}(\mathsf{V},-)+\mathcal{G}(\mathsf{V},-), G) \cong \mathbf{Grph}(\mathcal{G}(\mathsf{V},-), G)\times\mathbf{Grph}(\mathcal{G}(\mathsf{V},-), G) \cong G(\mathsf{V})\times G(\mathsf{V}),$$

with source and target functions given by the two projections $G(\mathsf{V})\times G(\mathsf{V}) \rightrightarrows G(\mathsf{V})$. It follows that $G^{\mathcal{G}(\mathsf{V},-)}$ is the complete graph on the vertices of G and that the induced map $G^{\mathcal{G}(\mathsf{V},-)} \to G$ is an isomorphism if and only if G is also complete. □

Proof of Lemma 2.23. Pushouts in $\mathbf{Grph} = [\mathcal{G}, \mathbf{Set}]$ are computed objectwise. For vertex-sets, we have the square

$$\begin{array}{ccccc} & & \{0,\dots,m+n-\ell\} & & \\ & \overset{p^m_{m+n-\ell}}{\nearrow} & & \overset{s^n_{m+n-\ell}}{\nwarrow} & \\ \{0,\dots,m\} & & & & \{0,\dots,n\} \\ & \underset{s^\ell_m}{\nwarrow} & & \underset{p^\ell_n}{\nearrow} & \\ & & \{0,\dots,\ell\}. & & \end{array}$$

Following the arrows around the left yields $i \mapsto i-\ell+m \mapsto i-\ell+m$, while following the arrows around the right yields $i \mapsto i \mapsto i-n+(m+n-\ell) = i-\ell+m$, so the square commutes. Moreover every arrow is injective, so the square above is equivalent to the square of subset inclusions

$$\begin{array}{ccccc} & & \{0,\dots,m+n-\ell\} & & \\ & \nearrow & & \nwarrow & \\ \{0,\dots,m\} & & & & \{m-\ell,\dots,m+n-\ell\} \\ & \nwarrow & & \nearrow & \\ & & \{m-\ell,\dots,m\}. & & \end{array}$$

As the bottom set is the intersection of the left and right sets, while the top set is their union, the square is a pushout.

Then for the entire square graphs to be a pushout, for every pair of vertices (i,j) in the lower graph (so $0 \le i,j \le \ell$), the set of edges in each graph whose source vertex is the image of i and whose target vertex is the image of j should be part of a pushout square. Indeed, every path graph has a unique edge $i \to j$ if $j-i=1$ and no edges $i \to j$ otherwise, and prefix and suffix maps preserve differences between vertices; so the square of edge-sets consists of either all singletons or all empty sets and is thus always a pushout. □

Proof of Corollary 2.24. We let $\overline{P}\colon \mathbf{I}_{\mathbb{Z}} \to \mathbf{Grph}$ denote the cosheaf of graphs equivalent to $P\colon \mathbf{Z} \to \mathbf{Grph}$ under the equivalence $\mathbf{Cu}_{\mathbb{Z}}(\mathbf{Grph}) \simeq [\mathbf{Z}, \mathbf{Grph}]$ from Proposition 2.3; by that result, $\overline{P}$ restricts to P along the inclusion $\mathbf{Z} \hookrightarrow \mathbf{I}_{\mathbb{Z}}$.

We will show by induction on $n \in \mathbb{N}$ that for all $t \in \mathbb{Z}$,

- $\overline{P}[t,t+n]$ and $\overline{P}[t,t+n+1]$ are path graphs;
- $\overline{P}$ sends $[t,t+n] \to [t,t+n+1]$ in $\mathbf{I}_{\mathbb{Z}}$ to a prefix map; and
- $\overline{P}$ sends $[t,t+n] \to [t-1,t+n]$ in $\mathbf{I}_{\mathbb{Z}}$ to a suffix map.

For $n = 0$, we have that $\overline{P}$ agrees with P on $[t-1,t] \leftarrow [t] \to [t,t+1]$ in $\mathbf{Z} \subseteq \mathbf{I}_{\mathbb{Z}}$, so the result follows by assumption. Then for $n > 0$, the cosheaf condition yields the following pushout square in **Grph**:

$$\begin{array}{ccccc} & & \overline{P}[t,t+n+1] & & \\ & \nearrow & \vee & \nwarrow & \\ \overline{P}[t,t+n] & & & & \overline{P}[t+1,t+n+1] \\ & \nwarrow & & \nearrow & \\ & & \overline{P}[t+1,t+n]. & & \end{array}$$

By induction, the three lower graphs are path graphs, the lower left arrow is a suffix map, and the lower right arrow is a prefix map, so by Lemma 2.23, the upper graph is also a path graph, the upper left arrow is a prefix map, and the upper right arrow is a suffix map, so the result follows. □