Syntactic linearity and Linear hyperdoctrines for second-order intuitionistic Linear Logic

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Extended Abstract

Introduction

The design of proof languages plays a central role in both logic and programming languages, particularly when reasoning about resource usage and algebraic computation. Linear Logic [7] provides a framework for such reasoning, but its standard proof-term calculi do not make linear properties explicit at the syntactic level, such as distributivity over addition or scalar multiplication. A linear calculus with syntactic linear properties can be useful in settings like quantum computing, where linearity is a fundamental constraint. In such a framework, syntactic linearity can be used to represent matrices and vectors directly as proof-terms.

This increase in expressiveness requires operations such as addition and scalar multiplication, which are typically absent from the proof language. In [3, 6] this challenge is addressed by introducing the $\mathcal{L}^{\mathcal{S}}$-calculus, a proof language for the non-exponential fragment of intuitionistic Linear Logic (IMALL) extended with syntactic constructs for addition ($\boldsymbol{+}$) and scalar multiplication ($\bullet$). This calculus has been extended to second-order Linear Logic [4], and a categorical model for its exponential fragment has been developed [5]. The present abstract focuses on a categorical model for the full calculus.

The $\mathcal{L}_2^{\mathcal{S}}$-calculus

The $\mathcal{L}_2^{\mathcal{S}}$-calculus [4] is a proof language for second-order intuitionistic Linear Logic, where propositions are generated by the following grammar:

$$A = X \mid \mathtt{1} \mid A \multimap A \mid A \otimes A \mid \top \mid \mathtt{0} \mid A \mathbin{\&} A \mid A \oplus A \mid !A \mid \forall X.A$$

Let $\mathcal{S}$ be a fixed semiring. The $\mathcal{L}_2^{\mathcal{S}}$-calculus extends the proof language with addition and scalar multiplication over $\mathcal{S}$. This extension includes replacing the proof-term $\star$ of proposition $\mathtt{1}$ with a family of proof-terms $a.\star$, one for each scalar a in $\mathcal{S}$. It also adds

$$
\begin{aligned}
(\!|1|\!) &= I & (\!|0|\!) &= \mathbf{0} \\
(\!|A \multimap B|\!) &= [(\!|A|\!), (\!|B|\!)] & (\!|A \,\&\, B|\!) &= (\!|A|\!) \oplus (\!|B|\!) \\
(\!|A \otimes B|\!) &= (\!|A|\!) \otimes (\!|B|\!) & (\!|A \oplus B|\!) &= (\!|A|\!) \oplus (\!|B|\!) \\
(\!|\top|\!) &= \mathbf{0} & (\!|!A|\!) &= !(\!|A|\!)
\end{aligned}
$$

Figure 1: Interpretation of propositions in the fragment without second order. The functor ! is the monoidal comonad in the Linear category, and $\mathbf{0}$ denotes the zero object.

inference rules for proof-terms of the form $t \boldsymbol{+} u$ and $a \bullet t$, without altering provability. To achieve cut-elimination, several commutation rules (between each connective and the new proof-term constructors $\boldsymbol{+}$ and $\bullet$) are added to the language.

Syntactic linearity

The $\mathcal{L}_2^{\mathcal{S}}$-calculus expresses linearity in a syntactic way: $t\ (u \boldsymbol{+} v)$ is observationally equivalent to $t\ u \boldsymbol{+} t\ v$, for any proof-term t of $A \multimap B$. Similarly, $t\ (a \bullet u)$ is equivalent to $a \bullet t\ u$. This linearity property makes it possible to represent vectors and matrices over $\mathcal{S}$ as proof-terms.

Denotational semantics: work in progress

A sound and adequate model for the fragment of the $\mathcal{L}_2^{\mathcal{S}}$-calculus without second order was defined [5] using a Linear category $(\mathcal{C}, \otimes, I, !)$ [1] with biproduct $\oplus$, together with a semiring monomorphism from $\mathcal{S}$ to $Hom(I, I)$. The interpretation of propositions as objects in this category is given in Figure 1. The monomorphism condition ensures completeness for the type 1. An alternative formulation can be given in terms of $\mathcal{S}$-semimodule-enriched Linear categories, leading to more compact proofs.

To account for second order, we define a hyperdoctrine [2] indexing $\mathcal{S}$-semimodule-enriched Linear categories, extending work by Maneggia [8] on Linear hyperdoctrines.

We consider a base category $\mathcal{C}$ with a distinguished terminal object U, such that every object is generated by finite products of U. We also consider the category $\mathbf{C}_!^{\mathcal{S}}\mathcal{AT}$ of $\mathcal{S}$-semimodule-enriched Linear categories with enriched functors. We define a hyperdoctrine via a functor $\mathcal{C}(-, U) : \mathcal{C}^{op} \to \mathbf{C}_!^{\mathcal{S}}\mathcal{AT}$, where for every n there is a functor $\forall_{U^n} : \mathcal{C}(U^n \times U, U) \to \mathcal{C}(U^n, U)$ that is a right adjoint. In this setting, a well-formed proposition is interpreted as an object in the category $\mathcal{C}(U^n, U)$, where n is the number of free type variables in it. The interpretation of $\forall X.A$ is then defined as $[\![\forall X.A]\!] = \forall_{U^n}([\![A]\!])$, where the application of the functor $\forall_{U^n}$ represents the binding of the variable X which is no longer free in $\forall X.A$.

Soundness of the model follows from the enrichment hypothesis on the functors that are morphisms of $\mathbf{C}_!^{\mathcal{S}}\mathcal{AT}$. Adequacy in this model depends on a semiring monomorphism condition similar to the one mentioned above, ensuring sufficient space such that all proofs of 1 are interpreted as different morphisms. We are currently working toward a better understanding and formulation of this condition.

References

[1] G.M. Bierman. On intuitionistic linear logic. Technical Report UCAM-CL-TR-346, University of Cambridge, Computer Laboratory, August 1994.

[2] Roy L. Crole. *Categories for types*. Cambridge University Press, 1993.

[3] Alejandro Díaz-Caro and Gilles Dowek. A linear linear lambda-calculus. *Mathematical Structures in Computer Science*, 34:1103–1137, 2024.

[4] Alejandro Díaz-Caro, Gilles Dowek, Malena Ivnisky, and Octavio Malherbe. A linear proof language for second-order intuitionistic linear logic. In George Metcalfe, Thomas Studer, and Ruy de Queiroz, editors, *Logic, Language, Information and Computation*, volume 14672 of *Lecture Notes in Computer Science*, pages 18–35. Springer, 2024.

[5] Alejandro Díaz-Caro, Malena Ivnisky, and Octavio Malherbe. An algebraic extension of intuitionistic linear logic: The $\mathcal{L}^s_!$-calculus and its categorical model. *Journal of Logic and Computation*, 35(8):exaf053, 2025.

[6] Alejandro Díaz-Caro and Octavio Malherbe. The sup connective in IMALL: A categorical semantics. To appear in TCS. Draft at `arXiv:2205.02142`, 2026.

[7] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[8] P. Maneggia. *Models of linear polymorphism*. PhD thesis, The University of Birmingham, UK, 2004.