Representability theory for multiactegories

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Monoidal categories are a classical structure with abundant applications across different areas of applied category theory, widely used to model compositionality. In some instances, a more general structure of a **multicategory**, or coloured operad, has been considered as a suitable language of compositional systems (cf. for instance [Lam89] or [FS19]).

While multicategories are strictly more general than monoidal categories, the notion of a representable multicategory provides an equivalent description of the latter in the context of multicategories. Besides monoidal categories, the theory of representabilty of multicategories allows one to additionally study for instance **closed categories** or **promonoidal categories**, all within a single context [Man09, DPS05].

In this talk, we present an ongoing work on representability theory for a slightly more complex structure of a **multiactegory**, or action of a multicategory. Starting with a multicategory $\mathbb{M}$, a multiactegory $\mathbb{A}$ has multihomsets of form $\mathbb{A}(a, \vec{m}; b)$ for each $a, b \in |\mathbb{A}|$ and a list $\vec{m}$ of objects in $|\mathbb{M}|$. In this setting, one can consider $\mathbb{A}$ to be representable in a suitable sense. Taking $\mathbb{M}$ to be representable as well, one may additionally impose a coherence between these two notions of representability. Furthermore, we are interested in (certain generalised) muliactegory morphisms of form $L : \mathbb{A} \to \mathbb{M}$ and define representability in the context of these.

We demonstrate that the theory of representable multiactegories and multiactegory morphisms provides an expressive framework in which one can capture numerous categorical structures arising in practice both in pure and applied category theory including (skew) actegories, skew warpings, skew monoidal categories, skew multicategories but also for example enriched categories.

As monoidal categories can be seen as a categorification of a monoid, so can **actegories** (actions of monoidal categories, or module categories) be seen as categorification of monoid actions. They have proven to be useful for modeling certain systems which call for a more complex structure than that of a monoidal category. In [FKMS25] for instance, actegories are used to characterise abstract syntax of languages with second-class sorts, such as Call-By-Value or Call-By-Push-Value λ-calculi. In [Gar18], tangent categories, which provide a category-theoretic setting for differential structures in geometry, algebra and computer science, were shown to be equivalent to certain actegories. Module categories furthermore play an integral role in categorification of representation theory, see for example [BSW20].

Actegories themselves can carry additional structure such as a **skew warping** (see [LS15]). In some examples the action comes equipped with a pair of adjoint functors between the acting category and the actegory, see for instance [SF18, slide 16]. These additional data then allow one to define a skew monoidal structure on the actegory [LS15, Pro25].

The definition of **skew monoidal categories** was motivated by the study of relative monads [ACU10] and characterisation of bialgebroids [Szl12] with applications to quantum categories. Analogous to the notion of multicategory, **skew multicategories** were defined in [BL18] together with a suitable notion of representability, called *left representability*, which induces a 2-equivalence with skew monoidal categories.

The relationship between the structures listed above may be illustrated as follows

skew monoidal categories

$\cup$

(skew) actegories $\supset$ skew warpings on actegories $\supset$ actegories with adjunction

Considering different variants of representability of multiactegories and their morphisms now allows us to characterise and study all the structures in the above diagram in the unified setting of multiactegories and their morphisms. As special cases we recover the left representability of [BL18] and the representability theorem of [SZ24].

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