Nuclearity and Trace in Monoidal Bicategories with Application to Extended CFTs: Extended Abstract

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We develop the theory of nuclear and trace 2-ideals in monoidal bicategories. We show that such 2-ideals are present in a certain bicategory of conformal cobordisms and in the bicategory of measured categories. Viewing the latter as a model of infinite dimensional 2-Hilbert spaces we give a definition of a once extended conformal field theory as a nuclear 2-functor.

1 Introduction

The theory of traces for monoidal categories is well appreciated and has many applications, notably in quantum information and in Geometry of Interaction models of linear logic. For instance, in the category of finite dimensional Hilbert spaces, traces allow one to calculate the state of a subsystem by removing the degrees of freedom associated to environment systems; while in the Geometry of Interaction, traces allow for a description of Girard's fundamental feedback law.

While the examples of traced categories abound, there also exist many categories which are almost but not quite traced, in particular, there are categories where some but not all morphisms permit a trace. This is the case with the category Hilb of Hilbert spaces and bounded linear maps and thus a number of weakenings of trace structure have been developed to formalise this. The difficulty posed by the class of traceable maps is that while they are typically closed under composition by any morphism in the larger category, the identity morphisms are generally not traceable. The notion of a *trace ideal* was introduced by Abramsky et al. [1] to formalise this: the traceable morphisms do not generally form a sub-category but only a two-sided ideal inside the overall category and this two-sided ideal can be shown to satisfy conditions (sliding, vanishing etc.) similar to those of a traced monoidal category. The category Hilb provides the prototypical example of a category permitting a trace ideal, in this case given by the trace class operators.

Abramsky et al. [1] also introduced nuclear ideals and showed that they are closely related to trace ideals. A nuclear ideal captures how close a monoidal category is to being compact closed: a morphism $f : a \to b$ in the nuclear ideal permits a transpose $\bar{f} : i \to \bar{a} \otimes b$. Much like for trace ideals, the identity morphism generally does not permit a transpose, and thus the nuclear morphisms only form a two-sided ideal. Any category with a nuclear ideal has an induced trace ideal given by those morphisms which factorise as the composition of two nuclear maps. Hilb has a nuclear ideal given by the Hilbert-Schmidt maps and the correspondence between nuclear and trace ideals recovers the known correspondence between trace class maps and Hilbert-Schmidt operators.

One application of trace ideals is to conformal field theories (CFTs). Roughly speaking a CFT is a functor from a category of cobordisms with conformal structure to Hilb,

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Z should assign trace class maps to the cobordisms and be monoidal, sending the disjoint union of manifolds to the tensor product of Hilbert spaces. Yet, there is a problem with this definition: $\mathsf{Cob}_{\mathrm{conf}}$ does not naïvely contain identity morphisms and if one artifically adds them their image under Z must necessarily be non-trace class (assuming an infinite dimensonal Hilbert space as the image of Z on the circle). Blute et al. [4] argued for a resolution to this problem by equipping $\mathsf{Cob}_{\mathrm{conf}}$ with a nuclear ideal given by the cobordisms with positive volume. Z now becomes a monoidal nuclear functor, the canonical structure preserving morphism between monoidal categories with nuclear ideals, and the desired properties of the CFT are inherent.

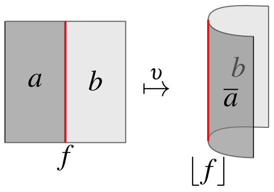
2 Main Results

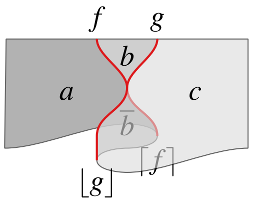
Our aim is to develop a 2-categorical version of the previous discussion coming both from a desire to understand trace and nuclear structures in monoidal bicategories and to develop extended CFTs.

2.1 Nuclear and Trace 2-Ideals

Traces for bicategories, known as *shadows*, have been introduced by Ponto [9] and furthered in work with Shulman [10]. A shadow for a bicategory consists of a family of functors $\langle\langle - \rangle\rangle_a : \mathscr{B}(a,a) \to \mathscr{T}$, one for each 0-cell a, taking the category of endo-cells of a a to a specified category $\mathscr{T}$ in which the traces are valued. These functors satisfy laws which weaken some of those familiar from traced monoidal categories: for instance sliding is now witnessed by a given invertible 2-cell.

The theory of shadows for *monoidal* bicategories has not yet been developed, nor has the theory of higher trace and nuclear ideals. Here we develop both. We do so by noting that to ask for an ideal is simply to ask for a sub-profunctor of the hom-profunctor. This allows us to abstract many of the conditions and by utilising the under-appreciated theory of pseudocoends [5], give a concise definition of nuclear and trace 2-ideals which subsumes the theory of shadows. A trace 2-ideal can be understood to give a class of traceable 1-cells in a bicategory where in general the identity 1-cells will not be traceable. Similarly, a nuclear 2-ideal can be understood to capture how close a bicategory is to being compact closed. A 1-cell $f : a \to b$ is nuclear when it possesses a transpose $\lfloor f \rfloor : i \to b \otimes \overline{a}$ (left hand below) together with 2-cells (right below) witnessing the snake/yanking equations.





2.2 Extended CFTs

Extended CFTs are intended to better capture the gluing properties of the path integral by breaking the manifolds into lower dimensional parts. This means that the domain and codomain of Z become higher categories with $\mathsf{Cob}_{\mathrm{conf}}$ containing, in principle, cobordisms between cobordisms between cobordisms etc, and Hilb replaced by a higher category of higher Hilbert spaces. We will consider just the 2-dimensional case which requires us to upgrade $\mathsf{Cob}_{\mathrm{conf}}$ to a bicategory of conformal cobordisms, Hilb to some category of 2-Hilbert spaces and deal with the structures that Z ought to preserve.

This programme has been partially carried out in [13, 6] with the desired bicategory $\mathsf{Cob}_{\mathrm{conf}}$ of conformal cobordisms developed in [13] and the codomain of Z taken to be the bicategory vN of von Neumann algebras and bimodules.

Z is still expected to assign trace class maps to the cobordisms and this is captured in [13] in terms of what they call the *adjunction transformations* (see Definition 2.1.1 therein). Here, we relate these transformations to the nuclear 2-ideals we develop. We show that $\mathsf{Cob}_{\mathrm{conf}}$ has a nuclear 2-ideal given by the cobordisms and furthering the defintion of Blute et al. [4] we define a generalised 2-extended CFT to be a nuclear 2-functor,

$$Z : \mathsf{Cob}_{\mathrm{conf}} \to \mathscr{B},$$

where $\mathscr{B}$ is a bicategory with nuclear 2-ideal. We then introduce the bicategory of measured categories Meas (closely related to Yetter's bicategory of measurable categories [15, 2]) as our model of infinite dimensional 2-Hilbert spaces and show that it possesses a nuclear 2-ideal given by a higher notion of Hilbert-Schmidt maps.

2.3 Infinite Dimensional 2-Hilbert Spaces

Part of the programme of defining extended CFTs requires the introduction of a notion of infinite dimensional 2-Hilbert space. While the finite dimensional case is well appreciated [8, 3] and has applications outside of topological and conformal field theories to provide higher semantics for quantum protocols [14, 11] and structures in quantum information [12], the infinite dimensional case is not well understood.

Here we introduce a bicategory inspired by the measurable categories of Yetter that can be understood to provide a notion of infinite dimensional 2-Hilbert space. Whereas all finite dimensional 2-Hilbert spaces are equivalent to a category of the form FHilb^n for $n \in \mathbb{N}$, our model of infinite dimensional 2-Hilbert spaces is based on categories of the form $\mathsf{Hilb}^{(X,\mu)}$ for some measure space (X,μ). The morphisms between infinite dimensional 2-Hilbert spaces can be thought of informally as a possibly infinite dimensional matrix of possibly infinite dimensional Hilbert spaces, generalising the morphisms of finite dimensional 2-Hilbert spaces. In fact, we show that this bicategory is equivalent to that of abelian von Neumann algebras and correspondences.

Theorem 2.4. *The bicategory* Meas *of measured categories is equivalent to the bicategory* avN *of abelian von Neumann algebras and* W^**-correspondences.*

This connects our notion with other suggestions for definitions of infinite dimensional 2-Hilbert spaces in terms of von Neumann algebras [13, 2] or equivalently in terms of Cauchy complete W^*-categories [7]. In comparison to those suggestions our bicategory, being a bicategory of "higher matrices" is closer in flavour to that of [14] and thus we hope this lays the groundwork to potential applications to infinite dimensional quantum information.

Acknowledgements

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Abstract. We develop the theory of nuclear and trace 2-ideals in monoidal bicategories. We show that such 2-ideals are present in a certain bicategory of conformal cobordisms and in the bicategory of measured categories. Viewing the latter as a model of infinite dimensional 2-Hilbert spaces we give a definition of a once extended conformal field theory as a nuclear 2-functor.

1. Introduction

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1.1. **Contributions.** Our aim is to develop a 2-categorical version of the previous discussion coming both from a desire to understand trace and nuclear structures in monoidal bicategories and to develop extended CFTs.

1.1.1. *Nuclear and Trace 2-Ideals.* Traces for bicategories, known as *shadows*, have been introduced by Ponto [22] and furthered in work with Shulman [23]. A shadow for a bicategory consists of a family of functors $\langle\langle - \rangle\rangle_a : \mathcal{B}(a,a) \to \mathcal{T}$, one for each 0-cell a, taking the category of endo-cells of a a to a specified category $\mathcal{T}$ in which the traces are valued. These functors satisfy laws which weaken some of those familiar from traced monoidal categories: for instance sliding is now witnessed by a given invertible 2-cell.

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1.1.2. *Biprofunctors and Pseudocoends.* The theory of biprofunctors and pseudocoends was developed in the 1970s by Bozapalides [7] but perhaps due to their PhD thesis being written in French and for many years difficult to retrieve, the theory seems to have been under-appreciated. Here we further that theory and apply it to develop the aforementioned notion of a 2-ideal. For maximum generality and forseeing future applications in which we might want additional structure on our functor Z such as continuity, we do this in the setting that our bicategories are themselves enriched in a monoidal bicategory. It is our hope that this might make all these notions more accessible to a wider audience. Along the way we also develop the theory of strength for biprofunctors on monoidal bicategories thereby categorifying a number of results of Pastro and Street [21].

1.1.3. *Extended CFTs.* Extended CFTs are intended to better capture the gluing properties of the path integral by breaking the manifolds into lower dimensional parts. This means that the domain and codomain of Z become higher categories with $\mathsf{Cob}_{\mathrm{conf}}$ containing, in principle, cobordisms between cobordisms between cobordisms etc, and Hilb replaced by a higher category of higher Hilbert spaces. We will consider just the 2-dimensional case which requires us to upgrade $\mathsf{Cob}_{\mathrm{conf}}$ to a bicategory of conformal cobordisms, Hilb to some category of 2-Hilbert spaces and deal with the structures that Z ought to preserve.

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where $\mathcal{B}$ is a bicategory with nuclear 2-ideal. We then take the bicategory of measurable categories Meas as our model of infinite dimensional 2-Hilbert spaces [25, 2] and show that it possesses a nuclear 2-ideal given by a higher notion of Hilbert-Schmidt maps.

2. Enriched Bicategories

The theory of bicategories enriched in a monoidal 2-category $\mathcal{V}$ dates back to the thesis of Bozapalides [7] and was generalised to monoidal bicategories in the theses of Carmody and Lack [9, 18]. We refer the reader to CITE for the precise definition of a monoidal bicategory, but let us fix some notation. We write $\boxtimes$ for the tensor and I for the unit 0-cell. The 1-cell components of the pseudo-natural equivalences witnessing the associativity and unitality of the tensor product $(\boxtimes, I)$ will be notated,

$$\alpha_{\mathcal{V}}^{ABC} : (A \boxtimes B) \boxtimes C \to A \boxtimes (B \boxtimes C), \qquad \lambda_{\mathcal{V}}^{A} : I \boxtimes A \to A, \qquad \rho_{\mathcal{V}}^{A} : A \boxtimes I \to A.$$

These equivalences are equipped with invertible modifications which we write as,

$$\begin{tikzcd}
& (AB)(CD) \arrow[dr, "\alpha_{\mathcal{V}}"] & \\
((AB)C)D \arrow[ur, "\alpha_{\mathcal{V}}"] \arrow[d, "\alpha_{\mathcal{V}}1"'] & \mathfrak{p}_{\mathcal{V}} \Uparrow & A(B(CD)) \\
(A(BC))D \arrow[rr, "\alpha_{\mathcal{V}}"'] & & A((BC)D) \arrow[u, "1\alpha_{\mathcal{V}}"']
\end{tikzcd}
\begin{tikzcd}
& AB & \\
(AI)B \arrow[ur, "\rho_{\mathcal{V}}1"] \arrow[rr, "\alpha_{\mathcal{V}}"'] & \mathfrak{m}_{\mathcal{V}} \Uparrow & A(IB) \arrow[ul, "1\lambda_{\mathcal{V}}"']
\end{tikzcd}
\begin{tikzcd}
& AB & \\
(AB)I \arrow[ur, "\rho_{\mathcal{V}}"] \arrow[rr, "\alpha_{\mathcal{V}}"'] & \mathfrak{r}_{\mathcal{V}} \Uparrow & A(BI) \arrow[ul, "1\rho_{\mathcal{V}}"']
\end{tikzcd}
\begin{tikzcd}
& AB & \\
(IA)B \arrow[ur, "\lambda_{\mathcal{V}}1"] \arrow[rr, "\alpha_{\mathcal{V}}"'] & \mathfrak{l}_{\mathcal{V}} \Uparrow & I(AB) \arrow[ul, "\lambda_{\mathcal{V}}"']
\end{tikzcd}$$

supressing the indices for the components and writing the tensor as concatenation. Let us now fix this monoidal bicategory $\mathcal{V}$ acting as our base of enrichment.

Definition 2.1. A **$\mathcal{V}$-bicategory** $\mathcal{B}$ consists of,

(1) a collection of 0-cells,
(2) for each pair of 0-cells (a, b), a 0-cell $\mathcal{B}(a, b)$ of $\mathcal{V}$,
(3) for every triple of 0-cells (a, b, c), a composition 1-cell of $\mathcal{V}$,

$$\mathcal{B}(b, c) \boxtimes \mathcal{B}(a, b) \xrightarrow{\circ_{abc}} \mathcal{B}(a, c)$$

(4) for every 0-cell a, a 1-cell of $\mathcal{V}$,

$$I \xrightarrow{j_a} \mathcal{B}(a, a)$$

(5) for every quadruple (a, b, c, d), an invertible 2-cell of $\mathcal{V}$ witnessing the associativity of composition,

$$\begin{tikzcd}
(\mathcal{B}(c, d) \boxtimes \mathcal{B}(b, c)) \boxtimes \mathcal{B}(a, b) \arrow[rr, "\alpha_{\mathcal{V}}"] \arrow[d, "\circ\boxtimes 1"'] & & \mathcal{B}(c, d) \boxtimes (\mathcal{B}(b, c) \boxtimes \mathcal{B}(a, b)) \arrow[d, "1\boxtimes\circ"] \\
\mathcal{B}(b, d) \boxtimes \mathcal{B}(a, b) \arrow[r, "\circ"'] & \mathcal{B}(a, d) & \mathcal{B}(c, d) \boxtimes \mathcal{B}(a, c) \arrow[l, "\circ"] \arrow[from=1-1, to=1-3, phantom, shift right=6, "\Longrightarrow"]
\end{tikzcd}$$

(6) for every pair (a, b), two invertible 2-cells of $\mathcal{V}$ witnessing the unitality of composition.

$$\begin{tikzcd}
I \boxtimes \mathcal{B}(b,a) \arrow[rr, "j_b \boxtimes 1"] \arrow[dr, "\lambda_{\mathcal{V}}"'] & \arrow[d, phantom, "\Downarrow"] & \mathcal{B}(b,b) \boxtimes \mathcal{B}(b,a) \arrow[dl, "\circ"] \\
& \mathcal{B}(b,a) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}(b,a) \boxtimes I \arrow[rr, "1 \boxtimes j_a"] \arrow[dr, "\rho_{\mathcal{V}}"'] & \arrow[d, phantom, "\Downarrow"] & \mathcal{B}(b,a) \boxtimes \mathcal{B}(a,a) \arrow[dl, "\circ"] \\
& \mathcal{B}(b,a) &
\end{tikzcd}$$

These data must satisfy a number of coherence conditions, see [7, 9, 18].

Example 2.2. A Cat-bicategory is (up to size issues) the usual notion of a bicategory.

Example 2.3. Taking $\mathcal{V} = \mathsf{Prof}$, the bicategory of categories, profunctors and natural transformations, gives the notion of a probicategory [13]. While bicategories are the horizontal categorification of monoidal categories, probicategories are the horizontal categorification of promonoidal categories [12, 11]. They possess a *virtual* composition of 1-cells so that $(b \xrightarrow{g} c) \circ (a \xrightarrow{f} b)$ is a presheaf $\mathcal{B}(a,c)^{\mathrm{op}} \to \mathsf{Set}$. The local presheaf bicategory $\widehat{\mathcal{B}}$ with the same 0-cells as $\mathcal{B}$ and hom-categories $\widehat{\mathcal{B}}(a,b) := [\mathcal{B}(a,b)^{\mathrm{op}}, \mathsf{Set}]$ is a closed bicategory under local Day convolution.

Example 2.4. Taking $\mathcal{V}$ to have discrete hom-categories so that $\mathcal{V}$ is the bicategory induced by adding only identity 2-cells to a monoidal category, gives the usual notion of an enriched category.

Definition 2.5. A **$\mathcal{V}$-pseudofunctor** $F : \mathcal{B} \to \mathcal{C}$ consists of,

(1) An assignment $F : \mathrm{Ob}\,\mathcal{B} \to \mathrm{Ob}\,\mathcal{C}$ of 0-cells of $\mathcal{B}$ to 0-cells of $\mathcal{C}$,
(2) For each pair (a, b), a 1-cell of $\mathcal{V}$,

$$\mathcal{B}(a,b) \to \mathcal{C}(Fa, Fb)$$

(3) For each triple (a, b, c), invertible 2-cells of $\mathcal{V}$,

$$\begin{tikzcd}
\mathcal{B}(b,c) \boxtimes \mathcal{B}(a,b) \arrow[r, "\circ"] \arrow[d, "F \boxtimes F"'] & \mathcal{B}(a,c) \arrow[d, "F"] \\
\mathcal{B}(Fb,Fc) \boxtimes \mathcal{B}(Fa,Fb) \arrow[r, "\circ"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{B}(Fa,Fb)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[rr, "j_{Fa}"] \arrow[dr, "j_a"'] & \arrow[d, phantom, "\Uparrow"] & \mathcal{B}(Fa,Fa) \\
& \mathcal{B}(a,a) \arrow[ur, "F"'] &
\end{tikzcd}$$

such that a series of coherence conditions hold [7, 9, 18].

Definition 2.6. Let $F, G : \mathcal{B} \to \mathcal{C}$ be $\mathcal{V}$-pseudofunctors. A **$\mathcal{V}$-pseudonatural transformation** $\eta : F \to G$ consists of,

- for each 0-cell a, a 1-cell $\eta_a : Fa \to Ga$ of $\mathcal{C}_0$
- for each pair (a,b), an invertible 2-cell of $\mathcal{V}$,

$$\begin{tikzcd}
\mathcal{B}(a,b) \arrow[r, "G"] \arrow[d, "F"'] & \mathcal{C}(Ga,Gb) \arrow[d, "\mathcal{C}(\eta_a,1)"] \\
\mathcal{C}(Fa,Fb) \arrow[r, "\mathcal{C}(1,\eta_b)"'] \arrow[ur, phantom, "\overset{\eta_{ab}}{\Nearrow}"] & \mathcal{C}(Fa,Gb)
\end{tikzcd}$$

such that a series of coherence conditions hold [7, 9, 18].

Definition 2.7. Let $\eta, \delta : F \to G$ be $\mathcal{V}$-pseudonatural transformations. A **$\mathcal{V}$-modification** $\mathfrak{m} : \eta \to \delta$ consists of a 2-cell $\mathfrak{m}_a : \eta_a \to \delta_a$ of $\mathcal{C}_0$ for each 0-cell a of $\mathcal{B}$ such that $(\mathcal{C}(1, \mathfrak{m}_b) \boxtimes 1) \circ \eta_{ab} = \delta_{ab} \circ (\mathcal{C}(\mathfrak{m}_a, 1) \boxtimes 1)$.

From now on we take our base of enrichment $\mathcal{V}$ to be a symmetric monoidal bicategory.

Definition 2.8. The **tensor product** $\mathcal{B} \boxtimes \mathcal{C}$ of $\mathcal{V}$-bicategories $\mathcal{B}$ and $\mathcal{C}$ is given by the $\mathcal{V}$-bicategory whose 0-cells (b, c) are pairs of 0-cells $b \in \mathcal{B}$ and $c \in \mathcal{C}$ and whose hom-objects are

$$(\mathcal{B} \boxtimes \mathcal{C})((b, c), (b', c')) = \mathcal{B}(b, b') \boxtimes \mathcal{C}(c, c').$$

The rest of the data is induced in the obvious way from that of $\mathcal{B}$ and $\mathcal{C}$ and it is a conceptually straightforward, though very lengthy task to check that this defines a $\mathcal{V}$-bicategory. See [7] for further discussion of the remaining data.

Definition 2.9. A $\mathcal{V}$-bicategory $\mathcal{B}$ is **monoidal** when it is equipped with:

- a $\mathcal{V}$-pseudofunctor,

  $$- \otimes - : \mathcal{B} \boxtimes \mathcal{B} \to \mathcal{B},$$

  and a chosen 0-cell i,
- $\mathcal{V}$-pseudonatural equivalences,

  $$\alpha : (a \otimes b) \otimes c \to a \otimes (b \otimes c), \quad \lambda : i \otimes a \to a, \quad \rho : a \otimes i \to a$$

- invertible $\mathcal{V}$-modifications

  $$\begin{tikzcd}
((ab)c)d \arrow[r, "\alpha"] \arrow[d, "\alpha 1"'] & (ab)(cd) \arrow[r, "\alpha"] \arrow[d, phantom, "\mathfrak{p}\Uparrow"] & a(b(cd)) \\
(a(bc))d \arrow[rr, "\alpha"'] & & a((bc)d) \arrow[u, "1\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& ab & \\
(ai)b \arrow[ur, "\rho 1"] \arrow[rr, "\alpha"'] & \mathfrak{m}\Uparrow & a(ib) \arrow[ul, "1\lambda"']
\end{tikzcd}$$

  $$\begin{tikzcd}
& ab & \\
(ia)b \arrow[ur, "\lambda 1"] \arrow[rr, "\alpha"'] & \mathfrak{l}\Uparrow & i(ab) \arrow[ul, "\lambda"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& ab & \\
(ab)i \arrow[ur, "\rho"] \arrow[rr, "\alpha"'] & \mathfrak{r}\Uparrow & a(bi) \arrow[ul, "1\rho"']
\end{tikzcd}$$

  satisfying a series of coherence diagrams given by the Stasheff polytopes.

3. Biprofunctors and Pseudocoends

Definition 3.1. Let $\mathcal{B}$ be a $\mathcal{V}$-bicategory. The **opposite** $\mathcal{V}$-bicategory $\mathcal{B}^{\mathrm{op}}$ has the same 0-cells as $\mathcal{B}$ and hom-objects $\mathcal{B}^{\mathrm{op}}(a, b) := \mathcal{B}(b, a)$. Composition is given by,

$$\mathcal{B}^{\mathrm{op}}(b, c) \boxtimes \mathcal{B}^{\mathrm{op}}(a, b) = \mathcal{B}(c, b) \boxtimes \mathcal{B}(b, a) \cong \mathcal{B}(b, a) \boxtimes \mathcal{B}(c, b) \xrightarrow{\circ} \mathcal{B}(c, a) = \mathcal{B}^{\mathrm{op}}(a, c),$$

and with identities the same as $\mathcal{B}$. The cells witnessing the essential associativity and unitality of composition are obvious.

Definition 3.2. Let $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{V}$-bicategories. A **$\mathcal{V}$-biprofunctor** $P : \mathcal{B} \nrightarrow \mathcal{C}$ is a $\mathcal{V}$-pseudofunctor $P : \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{B} \to \mathcal{V}$.

Biprofunctors have a well-defined composition given by a categorified version of the coend [7, 20] which we now work towards.

Definition 3.3. Let $P : \mathcal{B} \nrightarrow \mathcal{B}$ be a $\mathcal{V}$-biprofunctor. A **pseudocowedge** for P consists of a 0-cell d of $\mathcal{V}$ which we call the **apex**, a 1-cell of $\mathcal{V}$ for each 0-cell a,

$$P(a, a) \xrightarrow{w_a} d,$$

and an invertible 2-cell of $\mathcal{V}$ for each pair (a,b) of 0-cells,

$$\begin{tikzcd}
P(a,b)\boxtimes\mathcal{B}(b,a) \arrow[rr, "\sigma_{\mathcal{V}}"] \arrow[d, "\circ"'] & & \mathcal{B}(b,a)\boxtimes P(a,b) \arrow[d, "\circ"] \\
P(a,a) \arrow[dr, "w_a"'] \arrow[rr, phantom, "w_{ab}"] & & P(b,b) \arrow[dl, "w_b"] \\
& d &
\end{tikzcd} \tag{1}$$

such that the following composite 2-cell,

$$\begin{tikzcd}
& \mathcal{B}^a_b(P^c_a\mathcal{B}^b_c) \arrow[r, "\alpha_{\mathcal{V}}"] & (\mathcal{B}^a_bP^c_a)\mathcal{B}^b_c \arrow[dr, "\sigma_{\mathcal{V}}"] \arrow[dl, phantom, "\cong"] & \\
(P^c_a\mathcal{B}^b_c)\mathcal{B}^a_b \arrow[ur, "\sigma_{\mathcal{V}}"] \arrow[r, "\alpha_{\mathcal{V}}"] \arrow[d, "\circ\otimes 1"'] \arrow[dr, phantom, "\cong"] & P^c_a(\mathcal{B}^b_c\mathcal{B}^a_b) \arrow[r, "\sigma_{\mathcal{V}}"] \arrow[d, "1\otimes\circ"'] \arrow[dr, phantom, "\cong"] & (\mathcal{B}^b_c\mathcal{B}^a_b)P^c_a \arrow[r, "\alpha_{\mathcal{V}}"] \arrow[d, "\circ\otimes 1"] \arrow[dr, phantom, "\cong"] & \mathcal{B}^b_c(\mathcal{B}^a_bP^c_a) \arrow[d, "1\otimes\circ"] \\
P^b_a\mathcal{B}^a_b \arrow[dr, "\circ"'] & P^c_a\mathcal{B}^a_c \arrow[r, "\sigma_{\mathcal{V}}"] \arrow[d, "\circ"'] & \mathcal{B}^a_cP^c_a \arrow[d, "\circ"] & \mathcal{B}^b_cP^c_b \arrow[dl, "\circ"] \\
& P^a_a \arrow[d, "w_a"'] \arrow[r, phantom, "w_{ac}"] & P^c_c \arrow[dl, "w_c"] & \\
& d & &
\end{tikzcd} \tag{2}$$

is equal to the following 2-cell,

$$\begin{tikzcd}
(P^c_a\mathcal{B}^b_c)\mathcal{B}^a_b \arrow[rr, "\sigma_{\mathcal{V}}"] \arrow[d, "\circ\otimes 1"'] \arrow[drr, phantom, "\cong"] & & \mathcal{B}^a_b(P^c_a\mathcal{B}^b_c) \arrow[rr, "\alpha_{\mathcal{V}}"] \arrow[d, "1\otimes\circ"'] \arrow[drr, phantom, "\cong"] & & (\mathcal{B}^a_bP^c_a)\mathcal{B}^b_c \arrow[rr, "\sigma_{\mathcal{V}}"] \arrow[d, "\circ\otimes 1"] \arrow[drr, phantom, "\cong"] & & \mathcal{B}^b_c(\mathcal{B}^a_bP^c_a) \arrow[d, "1\otimes\circ"] \\
P^b_a\mathcal{B}^a_b \arrow[rr, "\sigma_{\mathcal{V}}"] \arrow[d, "\circ"'] & & \mathcal{B}^a_bP^b_a \arrow[dr, "\circ"'] & & P^c_b\mathcal{B}^b_c \arrow[rr, "\sigma_{\mathcal{V}}"] \arrow[dl, "\circ"] & & \mathcal{B}^b_cP^c_b \arrow[d, "\circ"] \\
P^a_a \arrow[drrr, "w_a"'] \arrow[rrr, phantom, "w_{ab}"] & & & P^b_b \arrow[d, "w_b"] \arrow[rrr, phantom, "w_{bc}"] & & & P^c_c \arrow[dlll, "w_c"] \\
& & & d & & &
\end{tikzcd} \tag{3}$$

and the following 2-cells are equal.

$$\begin{tikzcd}
& P^a_ai_{\mathcal{V}} \arrow[rr, "\sigma_{\mathcal{V}}"] \arrow[d, "1\otimes\eta_a"'] \arrow[ddl, bend right, "\rho_{\mathcal{V}}"'] \arrow[drr, phantom, "\cong"] & & i_{\mathcal{V}}P^a_a \arrow[d, "\eta_a\otimes 1"] \arrow[ddr, bend left, "\lambda_{\mathcal{V}}"] & \\
\arrow[r, phantom, "j_{aa}"] & P^a_a\mathcal{B}^a_a \arrow[rr, "\sigma_{\mathcal{V}}"] \arrow[dl, "\circ"] & & \mathcal{B}^a_aP^a_a \arrow[dr, "\circ"'] \arrow[r, phantom, "i_{aa}"] & {} \\
P^a_a \arrow[drr, "w_a"'] \arrow[rrrr, phantom, "w_{aa}"] & & & & P^a_a \arrow[dll, "w_a"] \\
& & d & &
\end{tikzcd}
=
\begin{tikzcd}
P^a_ai_{\mathcal{V}} \arrow[rr, "\sigma_{\mathcal{V}}"] \arrow[dr, "\rho_{\mathcal{V}}"'] & {} \arrow[d, phantom, "\cong"] & i_{\mathcal{V}}P^a_a \arrow[dl, "\lambda_{\mathcal{V}}"] \\
& P^a_a \arrow[d, "w_a"] & \\
& d &
\end{tikzcd} \tag{4}$$

Definition 3.4. Let w and w' be pseudocowedges for P both with apex d. A **modification** $m : w \to w'$ is a family of invertible 2-cells of $\mathcal{V}$,

$$\begin{tikzcd}
P(a,a) \arrow[r, bend left=50, "w_a"{name=U}] \arrow[r, bend right=50, "w'_a"'{name=D}] & d
\arrow[from=U, to=D, Rightarrow, "m_a"]
\end{tikzcd}$$

such that the following 2-cells are equal:

$$\begin{tikzcd}
P(a',a)\boxtimes\mathcal{B}(a,a') \arrow[rr, "\sim"] \arrow[d] & & \mathcal{B}(a,a')\boxtimes P(a',a) \arrow[d] \\
P(a',a') \arrow[dr, "w'_{a'}"'] & w'_{aa'} & P(a,a) \arrow[dl, bend right=30, "w'_a"'{name=A}] \arrow[dl, bend left=30, "w_a"{name=B}] \\
& d &
\arrow[from=A, to=B, phantom, "m_a"]
\end{tikzcd}
=
\begin{tikzcd}
P(a',a)\boxtimes\mathcal{B}(a,a') \arrow[rr, "\sim"] \arrow[d] & & \mathcal{B}(a,a')\boxtimes P(a',a) \arrow[d] \\
P(a',a') \arrow[dr, bend left=30, "w_{a'}"{name=A}] \arrow[dr, bend right=30, "w'_{a'}"'{name=B}] & w_{aa'} & P(a,a) \arrow[dl, "w_a"] \\
& d &
\arrow[from=A, to=B, phantom, "m_{a'}"]
\end{tikzcd}$$

Definition 3.5. A **pseudocoend** for P is a universal pseudocowedge, $(w, \fint^a P(a,a))$. This means that for any other pseudocowedge $(\bar{w}, d)$ there is a unique 1-cell $! : \fint^a P(a,a) \to d$ such that the cowedge components factor through ! up to an invertible 2-cell,

$$\begin{tikzcd}
P(a,a) \arrow[rr, "\bar{w}_a"] \arrow[dr, "w_a"'] & \cong & d \\
& \fint^a P(a,a) \arrow[ur, "!"'] &
\end{tikzcd}$$

and such that the following 2-cells are equal ensuring that the 2-isomorphisms witnessing the pseudocowedge factor via !.

$$\begin{tikzcd}
P(a',a)\boxtimes\mathcal{B}(a,a') \arrow[rr, "\sim"] \arrow[d] & & \mathcal{B}(a,a')\boxtimes P(a',a) \arrow[d] \\
P(a',a') \arrow[dr, "w_{a'}"] \arrow[ddr, bend right=40, "\bar{w}_{a'}"'] & \overset{w_{aa'}}{\cong} & P(a,a) \arrow[dl, "w_a"'] \arrow[ddl, bend left=40, "\bar{w}_a"] \\
\cong & \fint^a P(a,a) \arrow[d, "!"] & \cong \\
& d &
\end{tikzcd}
=
\begin{tikzcd}
P(a',a)\boxtimes\mathcal{B}(a,a') \arrow[rr, "\sim"] \arrow[d] & & \mathcal{B}(a,a')\boxtimes P(a',a) \arrow[d] \\
P(a',a') \arrow[dr, "\bar{w}_{a'}"'] & \overset{\bar{w}_{aa'}}{\cong} & P(a,a) \arrow[dl, "\bar{w}_a"] \\
& d &
\end{tikzcd}$$

Furthermore, any modification $m : \bar{w} \to \bar{\bar{w}}$ should factor uniquely through !, so that there exists a unique 2-isomorphism,

$$\begin{tikzcd}
\fint^a P(a,a) \arrow[r, bend left=50, "!\bar{w}"{name=U}] \arrow[r, bend right=50, "!\bar{\bar{w}}"'{name=D}] & d
\arrow[from=U, to=D, phantom, "\cong"]
\end{tikzcd}$$

such that

$$\begin{tikzcd}
P(a,a) \arrow[r, "w_a"] & \fint^a P(a,a) \arrow[r, bend left=50, "!\bar{w}"{name=U}] \arrow[r, bend right=50, "!\bar{\bar{w}}"'{name=D}] & d
\arrow[from=U, to=D, phantom, "\cong"]
\end{tikzcd}
=
\begin{tikzcd}
P(a,a) \arrow[r, bend left=50, "\bar{w}_a"{name=U}] \arrow[r, bend right=50, "\bar{\bar{w}}_a"'{name=D}] & d
\arrow[from=U, to=D, phantom, "\overset{m_a}{\cong}"]
\end{tikzcd}$$

Under reasonable assumptions on $\mathcal{V}$ (in particular that it is a bicomplete, closed, symmetric monoidal bicategory), it can be shown that the pseudocoend of P exists and the usual coend calculus can be extended to the setting of pseudocoends. In particular,

(1) Fubini's rule holds,

$$\fint^{a}\fint^{b} P(a,a,b,b) \cong \fint^{ab} P(a,a,b,b) \cong \fint^{b}\fint^{a} P(a,a,b,b)$$

(2) the ninja Yoneda lemma holds,

$$F- \cong \fint^{a} \mathcal{B}(-,a) \boxtimes Fa, \qquad F- \cong \fint_{a} \mathcal{V}(\mathcal{B}(a,-), Fa)$$

for any pseudofunctor $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$,

(3) the hom-biprofunctor preserves bilimits in each variable,

$$\mathcal{C}\left(c, \fint_{b} P(b,b)\right) \cong \fint_{b} \mathcal{C}(c, P(b,b)), \qquad \mathcal{C}\left(\fint^{b} P(b,b), c\right) \cong \fint_{b} \mathcal{C}(P(b,b), c)$$

for any $P : \mathcal{B}^{\mathrm{op}} \boxtimes \mathcal{B} \to \mathcal{C}$ for which the pseudocoend exists in $\mathcal{C}$.

4. Shadows

In [23] the notion of a shadow for a bicategory was introduced. In this section we will give a more concise definition, demonstrating that the coherence data can be captured by those of a pseudocoend. This will act as a stepping stone to traces in monoidal bicategories.

Definition 4.1. Let $\mathcal{B}$ be a bicategory. A **shadow** for $\mathcal{B}$ consists of a designated category $\mathcal{T}$ and a functor for each 0-cell a,

$$\langle\langle - \rangle\rangle : \mathcal{B}(a,a) \to \mathcal{T},$$

equipped with natural isomorphisms $\theta : \langle\langle gf \rangle\rangle \cong \langle\langle gf \rangle\rangle$ where $f : a \to b$ and $g : b \to a$, such that the following diagrams commute whenever they make sense.

$$\begin{tikzcd}
\langle\langle (hg)f \rangle\rangle \arrow[r, "\theta"] \arrow[d, "\langle\langle \mathfrak{a} \rangle\rangle"'] & \langle\langle f(hg) \rangle\rangle \arrow[r, "\langle\langle \mathfrak{a} \rangle\rangle"] & \langle\langle (fh)g \rangle\rangle & & \langle\langle f1 \rangle\rangle \arrow[r, "\theta"] \arrow[dr, "\langle\langle \mathfrak{r} \rangle\rangle"'] & \langle\langle 1f \rangle\rangle \arrow[d, "\langle\langle \mathfrak{l} \rangle\rangle"] \\
\langle\langle h(gf) \rangle\rangle \arrow[r, "\theta"'] & \langle\langle (gf)h \rangle\rangle \arrow[r, "\langle\langle \mathfrak{a} \rangle\rangle"'] & \langle\langle g(fh) \rangle\rangle \arrow[u, "\theta"'] & & & \langle\langle f \rangle\rangle
\end{tikzcd} \tag{5}$$

Remark. In comparison to [23] we do not ask for the other identity involving the units.

Proposition 4.2. *Let $\mathcal{B}$ be a bicategory. To give a shadow for $\mathcal{B}$ is to give a functor $\fint^{a} \mathcal{B}(a,a) \to \mathcal{T}$.*

Proof. By precomposing with the coprojections for the pseudocoend we get functors,

$$\mathcal{B}(a,a) \xrightarrow{\mathrm{copr}_a} \fint^{a} \mathcal{B}(a,a) \to \mathcal{T}$$

which we take to be the required functors $\langle\langle - \rangle\rangle$. The natural isomorphism θ is recovered by unpacking the higher dimensional data for the pseudocoend. Firstly, when $\mathcal{V} = \mathsf{Cat}$, the pseudocowedge

2-cell (1) yields the following 2-cell.

$$\begin{tikzcd}
\mathcal{B}(a,b) \arrow[r, "\mathcal{B}(1,g)"] \arrow[d, "\mathcal{B}(g,1)"'] \arrow[dr, phantom, "w"] & \mathcal{B}(a,a) \arrow[d, "\mathrm{copr}_a"] \arrow[ddr, bend left, "\langle\langle - \rangle\rangle"] & \\
\mathcal{B}(b,b) \arrow[r, "\mathrm{copr}_b"'] \arrow[drr, bend right, "\langle\langle - \rangle\rangle"'] & \int^a \mathcal{B}(a,a) \arrow[dr] & \\
 & & \mathcal{T}
\end{tikzcd}$$

Taking $f \in \mathcal{B}(a,b)$ and chasing this diagram gives the isomorphism $\theta : \langle\langle gf \rangle\rangle \cong \langle\langle fg \rangle\rangle$ natural in f. The naturality of the cowedge 2-cell (1) yields the following diagram,

$$\begin{tikzcd}
\mathcal{B}(a,b) \arrow[r, "\mathcal{B}(1,g')"] \arrow[d, bend right=50, "\mathcal{B}(g,1)"'] \arrow[d, bend left=50, "\mathcal{B}(g',1)"] \arrow[d, phantom, "\mathcal{B}(\eta,1)"] \arrow[dr, phantom, "w"] & \mathcal{B}(a,a) \arrow[d, "\mathrm{copr}_a"] & & \mathcal{B}(a,b) \arrow[r, bend left, "\mathcal{B}(1,g')"] \arrow[r, bend right, "\mathcal{B}(1,g)"'] \arrow[r, phantom, "\mathcal{B}(1,\eta)"] \arrow[d, "\mathcal{B}(g,1)"'] \arrow[dr, phantom, "w"] & \mathcal{B}(a,a) \arrow[d, "\mathrm{copr}_a"] \\
\mathcal{B}(b,b) \arrow[r, "\mathrm{copr}_b"'] & \int^a \mathcal{B}(a,a) \arrow[from=1-2, to=2-4, phantom, "="] & & \mathcal{B}(b,b) \arrow[r, "\mathrm{copr}_b"'] & \int^a \mathcal{B}(a,a)
\end{tikzcd}$$

which gives naturality of θ in g.

Now we will recover the coherences (5) from those of the pseudocoend. By the equality of (2) and (3) we have,

$$\begin{tikzcd}[column sep=small]
 & & \int^a \mathcal{B}(a,a) & & & & & \int^a \mathcal{B}(a,a) & & \\
\mathcal{B}(c,c) \arrow[urr, "\mathrm{copr}_c"] & w & \mathcal{B}(b,b) \arrow[u, "\mathrm{copr}_b"] & w & \mathcal{B}(a,a) \arrow[ull, "\mathrm{copr}_a"'] & = & \mathcal{B}(c,c) \arrow[ur, "\mathrm{copr}_c"] & w & \mathcal{B}(a,a) \arrow[ul, "\mathrm{copr}_a"'] & \\
 & \mathcal{B}(c,b) \arrow[ul, "\mathcal{B}(1,g)"] \arrow[ur, "\mathcal{B}(g,1)"] & \mathfrak{a} & \mathcal{B}(b,a) \arrow[ul, "\mathcal{B}(1,f)"'] \arrow[ur, "\mathcal{B}(f,1)"'] & & & \mathcal{B}(c,b) \arrow[u, "\mathcal{B}(1,g)"] & & \mathcal{B}(b,a) \arrow[u, "\mathcal{B}(f,1)"'] & \\
 & & \mathcal{B}(c,a) \arrow[ul, "\mathcal{B}(1,f)"] \arrow[ur, "\mathcal{B}(g,1)"'] & & & & & \mathcal{B}(c,a) \arrow[ul, "\mathcal{B}(1,f)"] \arrow[ur, "\mathcal{B}(g,1)"'] \arrow[uul, bend left, "\mathcal{B}(1,gf)"] \arrow[uur, bend right, "\mathcal{B}(gf,1)"'] & &
\end{tikzcd}$$

which recovers the hexagon of (5). Finally, by (4) we have,

$$\begin{tikzcd}
\mathcal{B}(a,a) \arrow[r, bend left, "1"] \arrow[r, bend right, "\mathcal{B}(1,1)"'] & \mathcal{B}(a,a) \arrow[r, "\mathrm{copr}_a"] & \int^a \mathcal{B}(a,a) & = & \mathcal{B}(a,a) \arrow[r, bend left, "1"] \arrow[r, bend right, "\mathcal{B}(1,1)"'] \arrow[dr, bend right, "\mathcal{B}(1,1)"'] & \mathcal{B}(a,a) \arrow[r, "\mathrm{copr}_a"] & \int^a \mathcal{B}(a,a) \\
 & & & & & \mathcal{B}(a,a) \arrow[ur, "\mathrm{copr}_a"'] &
\end{tikzcd}$$

recovering the triangle of (5). □

5. Trace 2-Ideals

Now we categorify the trace ideals of Abramsky et al. [1]. Let us begin by recalling the definition of a trace ideal.

Definition 5.1. Let $\mathcal{C}$ be a symmetric monoidal category. A **trace ideal** T for $\mathcal{C}$ consists of a specified subset $T(a,a) \subseteq \mathcal{C}(a,a)$ and a function $\mathrm{tr}_a : T(a,a) \to \mathcal{C}(i,i)$ for each object a such that,

(1) $T(a,a)$ is a two-sided ideal in $\mathcal{C}(a,a)$,
(2) if $f : a \to b$ and $g : b \to a$ are such that $gf \in T(a,a)$, then $fg \in T(b,b)$ and $\mathrm{tr}_a(gf) = \mathrm{tr}_b(fg)$,
(3) if $f \in T(a,a)$ and $g \in T(b,b)$, then $f \otimes g \in T(a \otimes b, a \otimes b)$ and $\mathrm{tr}_{a\otimes b}(f \otimes g) = \mathrm{tr}_a f \otimes \mathrm{tr}_b g$,
(4) $T(i,i) = \mathcal{C}(i,i)$ and $\mathrm{tr}_i\, s = s$ for each $s : i \to i$. If $f \in T(a,a)$ then $\mathrm{tr}_{a\otimes i}(f \otimes 1_i) = \mathrm{tr}_a f$.

Note that a trace ideal T is only defined on the diagonal components $T(a,a)$ but we can always extend it to the non-diagonal components by freely adding all compositions with morphisms in $\mathcal{C}$, that is, considering the profunctor freely generated by the ideal. This does not add anything to the diagonal components.

Lemma 5.2. *Every trace ideal T permits an extension to the non-diagonal components such that it becomes a profunctor $T : \mathcal{C} \nrightarrow \mathcal{C}$.*

Proof. Define $T'(a,c) = \{k \in \mathcal{C}(a,c) \mid \exists b \in \mathcal{C}, g \in T(b,b), f \in \mathcal{C}(a,b), h \in \mathcal{C}(b,c) \text{ with } k = hgf\}$. It is clear that T' is closed under composition by morphisms in $\mathcal{C}$ and so is a profunctor $\mathcal{C} \nrightarrow \mathcal{C}$. Consider $k \in T'(a,a)$ which by definition permits a decomposition as $k = hgf$ with $f \in \mathcal{C}(a,b)$, $g \in T(b,b)$ and $h \in \mathcal{C}(b,c)$. Note that $(fh)g \in T(b,b)$ since $T(b,b)$ is a two-sided ideal in $\mathcal{C}(b,b)$, so that $(fh)g$ permits a trace. By axiom (2), we have $k = hgf \in T(a,a)$ and we can conclude that $T(a,a) = T'(a,a)$ for each a. ☐

With this straightforward observation in place we can give a streamlined presentation of the data for a trace ideal which will be easier to generalise to bicategories.

Proposition 5.3. *Let $\mathcal{C}$ be a symmetric monoidal category. A trace ideal T for $\mathcal{C}$ consists of a monoidal sub-profunctor of the hom $T(-,-) \hookrightarrow \mathcal{C}(-,-)$ and a function $\mathrm{tr} : \int^a T(a,a) \to \mathcal{C}(i,i)$ such that the following diagram commutes.*

$$\begin{tikzcd}
\int^a T(a,a) \times \int^b T(b,b) \arrow[r, "\int \otimes"] \arrow[d, "\mathrm{tr} \times \mathrm{tr}"'] & \int^{ab} T(a \otimes b, a \otimes b) \arrow[d, "\mathrm{tr}"] \\
\mathcal{C}(i,i) \times \mathcal{C}(i,i) \arrow[r, "\otimes"'] & \mathcal{C}(i,i)
\end{tikzcd} \tag{6}$$

Proof. Suppose T is a trace ideal. By Lemma 5.2 we know that we can extend T to a profunctor $\mathcal{C} \nrightarrow \mathcal{C}$ whose diagonal components are the specified subsets of the original T. Since we have the dinaturality axiom (2), the universal property of the coend implies that there exists a unique function $\mathrm{tr} : \int^a T(a,a) \to \mathcal{C}(i,i)$ coming from the family tr_a.

Monoidality of T means that there is a natural transformation $\otimes : T(a,b) \times T(c,d) \to T(a\otimes c, b\otimes d)$ which together with square (6) recovers axiom (3). Monoidality of T also implies the existence of an arrow $\{*\} \to T(i,i)$ picking out a specified morphism $i \to i$ which must be the identity 1_i by chasing the coherence diagrams for the monoidal natural transformation $T(-,-) \hookrightarrow \mathcal{C}(-,-)$. Therefore $T(i,i) = \mathcal{C}(i,i)$ because it is a two-sided ideal in $\mathcal{C}(i,i)$ containing the identity. ☐

It it now conceptually straightforward to generalise trace ideals to monoidal bicategories.

Definition 5.4. Let $\mathcal{B}$ be a monoidal $\mathcal{V}$-bicategory. A **trace 2-ideal** for $\mathcal{B}$ consists of a monoidal sub-biprofunctor of the hom $T(-,-) \hookrightarrow \mathcal{B}(-,-)$ equipped with a functor $\mathrm{tr} : \fint^a T(a,a) \to \mathcal{B}(i,i)$ and an intertible 2-cell,

$$\begin{tikzcd}
\fint^a T(a,a) \boxtimes \fint^b T(b,b) \arrow[r, "\fint \otimes_T"] \arrow[d, "\mathrm{tr} \boxtimes \mathrm{tr}"'] & \fint^{ab} T(a \otimes b, a \otimes b) \arrow[r, "\otimes_{\mathcal{B}}"] \arrow[d, phantom, "\mathfrak{n}"] & \fint^a T(a,a) \arrow[d, "\mathrm{tr}"] \\
\mathcal{B}(i,i) \boxtimes \mathcal{B}(i,i) \arrow[r, "\otimes_{\mathcal{B}}"'] & \mathcal{B}(i \otimes i, i \otimes i) \arrow[r, "\rho"'] & \mathcal{B}(i,i)
\end{tikzcd}$$

such that the following 2-cell,

$$\begin{tikzcd}
(\int^{a} T^{a}_{a} \int^{b} T^{b}_{b}) \int^{c} T^{c}_{c} \arrow[r, "\otimes_{\mathcal{T}} 1"] \arrow[d, "(\mathrm{tr}\,\mathrm{tr})1"'] & \int^{ab} T^{ab}_{ab} \int^{c} T^{c}_{c} \arrow[r, "\otimes_{\mathcal{B}} 1"] \arrow[d, phantom, "\mathfrak{n}1"] & \int^{a} T^{a}_{a} \int^{c} T^{c}_{c} \arrow[r, "\otimes_{\mathcal{T}}"] \arrow[d, "\mathrm{tr}\,1"] & \int^{ac} T^{ac}_{ac} \arrow[r, "\otimes_{\mathcal{B}}"] & \int^{a} T^{a}_{a} \arrow[dd, "\mathrm{tr}"] \\
(\mathcal{B}^{i}_{i} \mathcal{B}^{i}_{i}) \int^{c} T^{c}_{c} \arrow[r, "\otimes_{\mathcal{B}} 1"] \arrow[d, "(11)\,\mathrm{tr}"'] \arrow[dr, phantom, "\cong"] & \mathcal{B}^{ii}_{ii} \int^{c} T^{c}_{c} \arrow[r, "\rho 1"] \arrow[d, "1\,\mathrm{tr}"] \arrow[dr, phantom, "\cong"] & \mathcal{B}^{i}_{i} \int^{c} T^{c}_{c} \arrow[d, "1\,\mathrm{tr}"] & \mathfrak{n} & \\
(\mathcal{B}^{i}_{i} \mathcal{B}^{i}_{i}) \mathcal{B}^{i}_{i} \arrow[r, "\otimes_{\mathcal{B}} 1"'] & \mathcal{B}^{ii}_{ii} \mathcal{B}^{i}_{i} \arrow[r, "\rho 1"'] & \mathcal{B}^{i}_{i} \mathcal{B}^{i}_{i} \arrow[r, "\otimes_{\mathcal{B}}"'] & \mathcal{B}^{ii}_{ii} \arrow[r, "\rho"'] & \mathcal{B}^{i}_{i}
\end{tikzcd}$$

is equal to following 2-cell (up to composing with some canonical tensorator and associator 2-cells on the boundary).

$$\begin{tikzcd}
\int^{a} T^{a}_{a} (\int^{b} T^{b}_{b} \int^{c} T^{c}_{c}) \arrow[r, "1\otimes_{\mathcal{T}}"] \arrow[d, "1(\mathrm{tr}\,\mathrm{tr})"'] & \int^{a} T^{a}_{a} \int^{bc} T^{bc}_{bc} \arrow[r, "1\otimes_{\mathcal{B}}"] \arrow[d, phantom, "1\mathfrak{n}"] & \int^{a} T^{a}_{a} \int^{c} T^{c}_{c} \arrow[r, "\otimes_{\mathcal{T}}"] \arrow[d, "1\,\mathrm{tr}"] & \int^{ac} T^{ac}_{ac} \arrow[r, "\otimes_{\mathcal{B}}"] & \int^{a} T^{a}_{a} \arrow[dd, "\mathrm{tr}"] \\
\int^{a} T^{a}_{a} (\mathcal{B}^{i}_{i} \mathcal{B}^{i}_{i}) \arrow[r, "1\otimes_{\mathcal{B}}"] \arrow[d, "\mathrm{tr}(11)"'] \arrow[dr, phantom, "\cong"] & \int^{a} T^{a}_{a} \mathcal{B}^{ii}_{ii} \arrow[r, "1\rho"] \arrow[d, "\mathrm{tr}\,1"] \arrow[dr, phantom, "\cong"] & \int^{a} T^{a}_{a} \mathcal{B}^{i}_{i} \arrow[d, "\mathrm{tr}\,1"] & \mathfrak{n} & \\
\mathcal{B}^{i}_{i} (\mathcal{B}^{i}_{i} \mathcal{B}^{i}_{i}) \arrow[r, "1\otimes_{\mathcal{B}}"'] & \mathcal{B}^{i}_{i} \mathcal{B}^{ii}_{ii} \arrow[r, "1\rho"'] & \mathcal{B}^{i}_{i} \mathcal{B}^{i}_{i} \arrow[r, "\otimes_{\mathcal{B}}"'] & \mathcal{B}^{ii}_{ii} \arrow[r, "\rho"'] & \mathcal{B}^{i}_{i}
\end{tikzcd}$$

Example 5.5. Every monoidal $\mathcal{V}$-bicategory with a shadow has a trace 2-ideal defined on the whole bicategory.

6. Nuclear 2-Ideals

Now we recall the definition of a nuclear ideal [1], recasting it our profunctorial language. For this we also need the notion of a tensored $*$-category.

Definition 6.1 ([1])**.** A monoidal category $\mathcal{C}$ is a **$*$-category** when it is equipped with an identity on objects and involutive **dagger** $(-)^{\dagger} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ satisfying $(f \otimes g)^{\dagger} = f^{\dagger} \otimes g^{\dagger}$ and a **conjugation** $\overline{(-)} : \mathcal{C} \to \mathcal{C}$ equipped with natural isomorphisms,

$$\overline{\overline{a}} \cong a, \qquad \overline{a \otimes b} \cong \overline{a} \otimes \overline{b}, \qquad \overline{i} \cong i,$$

such that the following diagram commutes for any $f : i \to i$.

$$\begin{tikzcd}
i \arrow[r, "f^{\dagger}"] \arrow[d, "\cong"'] & i \arrow[d, "\cong"] \\
\overline{i} \arrow[r, "\overline{f}"'] & \overline{i}
\end{tikzcd}$$

Remark. The dagger was traditionally known as the $*$, hence the name $*$-category, but we have decided to use the more modern notation † for this functor.

Definition 6.2. Let $\mathcal{C}$ be a monoidal $*$-category. A **nuclear ideal** for $\mathcal{C}$ consists of a specified subset $N(a, b) \subseteq \mathcal{C}(a, b)$ for each pair of objects a and b, and a natural isomorphism $\upsilon : N(a^{\dagger}, b) \cong \mathcal{C}(i, \overline{a} \otimes b)$ such that,

(1) N forms a 2-sided ideal in $\mathcal{C}$, that is, it is closed under composition on either side by morphisms of $\mathcal{C}$. This makes N a profunctor $\mathcal{C} \nrightarrow \mathcal{C}$.
(2) N is closed under $\otimes$, $\overline{(-)}$ and $(-)^{\dagger}$. The former makes N a *monoidal* profunctor.

(3) υ is a *monoidal* natural transformation so that the following diagrams commute,

$$\begin{tikzcd}
N(a,b)\times N(c,d) \arrow[rr, "\otimes"] \arrow[d, "\upsilon\times\upsilon"'] & & N(a\otimes c, b\otimes d) \arrow[d, "\upsilon"] \\
\mathcal{C}(i,\overline{a}\otimes b)\times\mathcal{C}(i,\overline{c}\otimes d) \arrow[dr, "\otimes"'] & & \mathcal{C}(i,(\overline{a\otimes c})\otimes(b\otimes d)) \\
& \mathcal{C}(i\otimes i,(\overline{a}\otimes b)\otimes(\overline{c}\otimes d)) \arrow[ur, "\cong"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \{*\} \arrow[dl] \arrow[dr] & \\
N(i,i) \arrow[rr, "\upsilon"'] & & \mathcal{C}(i,\overline{i}\otimes i)
\end{tikzcd}$$

the latter diagram implying that the transpose of a map $f : i \to a$ is given by composition with the canonical map $a \to \overline{i}\otimes a$.

(4) Transposition plays well with the conjugation and dagger so that $\overline{\upsilon(f)} = \upsilon(f^{\dagger}) = \sigma\circ\upsilon(\overline{f})$.

(5) The following diagram commutes,

$$\begin{tikzcd}
\int^{b} N(a,b)\times N(b,c) \arrow[rr, "\int\upsilon'\times\upsilon"] \arrow[dr, "\circ"'] & & \int^{b} N(a\otimes\overline{b},i)\times N(i,\overline{b}\otimes c) \arrow[dl, "\circ"] \\
& N(a,c) &
\end{tikzcd}$$

where υ' is the natural isomorphism $N(a,b)\xrightarrow{\dagger}N(b,a)\xrightarrow{\upsilon}N(i,\overline{b}\otimes a)\xrightarrow{\dagger}N(\overline{b}\otimes a,i)\xrightarrow{\cong}N(a\otimes\overline{b},i)$ given by conjugating υ by the dagger; the arrow $\circ : \int^{b} N(a,b)\times N(b,c)\to N(a,c)$ is given by composition in $\mathcal{C}$; and the arrow $\circ : \int^{b} N(a\otimes\overline{b},i)\times N(i,\overline{b}\otimes c)\to N(a,c)$ is given by composing along $\overline{b}$ in the obvious way in $\mathcal{C}$.

Definition 6.3. A monoidal $\mathcal{V}$-bicategory $\mathcal{B}$ is a $*$**-bicategory** when it is equipped with,

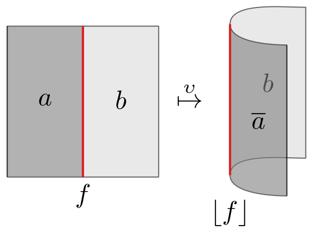
(1) an identity on 0-cells and strictly involutive dagger $(-)^{\dagger} : \mathcal{B}^{\mathrm{op}}\to\mathcal{B}$ which is strictly monoidal on 1-cells $(f\otimes g)^{\dagger} = f^{\dagger}\otimes g^{\dagger}$ and 2-cells $(\alpha\otimes\beta)^{\dagger} = \alpha^{\dagger}\otimes\beta^{\dagger}$,

(2) a conjugation $\overline{(-)} : \mathcal{B}\to\mathcal{B}$ with $\mathcal{V}$-natural isomorphisms $\overline{\overline{a}}\cong a$, $\overline{a\otimes b}\cong\overline{a}\otimes\overline{b}$ and $\overline{i}\cong i$.

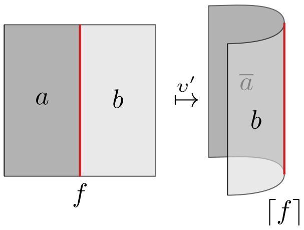
Definition 6.4. Let $\mathcal{B}$ be a monoidal $*$-bicategory. A **nuclear 2-ideal** for $\mathcal{B}$ consists of a monoidal sub-biprofunctor of the hom $N(-,-)\hookrightarrow\mathcal{B}(-,-)$ equipped with a monoidal pseudonatural $\mathcal{V}$-isomorphism,

$$\upsilon : N(a^{\dagger},b)\cong\mathcal{B}(i,\overline{a}\otimes b)$$

which we can picture acting like



We ask that N is closed under $(-)^\dagger$ and $\overline{(-)}$ so there is also a pseudonatural $\mathcal{V}$-isomorphism acting like

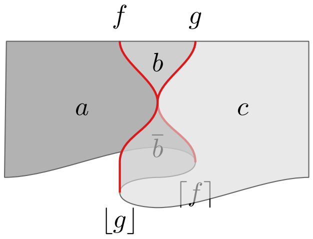


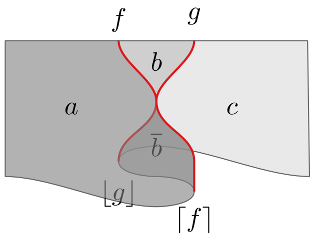
Furthermore we require invertible 2-cells,

$$\begin{tikzcd}
\int^b N(a,b)\times N(b,c) \arrow[rr, "\int \upsilon'\times\upsilon"] \arrow[ddr, "\circ_b"'] & & \int^b N(a\otimes\bar{b},i)\times N(i,\bar{b}\otimes c) \arrow[ddl, "\circ_{\bar{b}}"] \\
& \mathfrak{s} & \\
& N(a,c) &
\end{tikzcd}$$

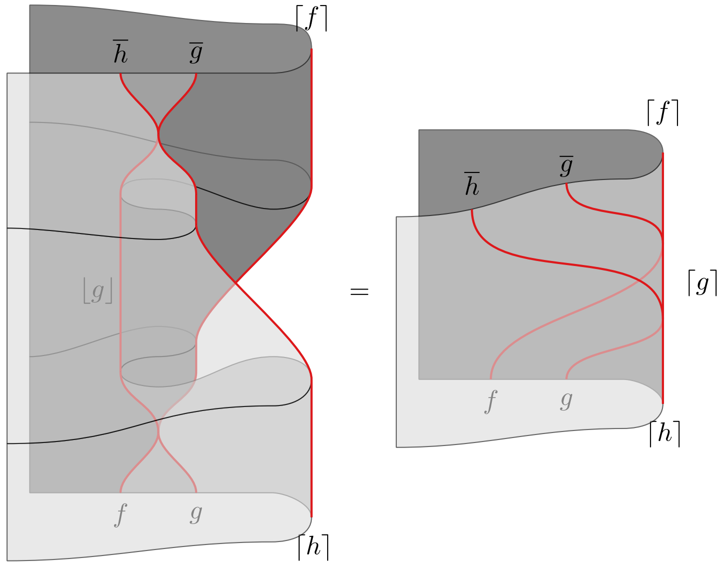
$$\begin{tikzcd}
\int^b N(a,b)\times N(b,c) \arrow[rr, "\int \upsilon'\times\upsilon"] \arrow[ddr, "\circ_b"'] & & \int^b N(\bar{b}\otimes a,i)\times N(i,c\otimes\bar{b}) \arrow[ddl, "\circ_{\bar{b}}"] \\
& \mathfrak{t} & \\
& N(a,c) &
\end{tikzcd}$$

which for each pair of nuclear morphisms f and g gives 2-cells witnessing the snake/yanking equations.





These 2-cells need to satisfy versions of the swallowtail equations depicted in the following diagram



where the left hand side is a composite of the snake 2-cells and the tensorator and the right hand side is a composite of 2-cells arising from the pseudonatural isomorphism υ. For instance, the first 2-cell (reading upwards) comes from considering

$$\begin{tikzcd}
N(c,d) \arrow[r, "N(g,1)"] \arrow[d, "\upsilon"'] \arrow[dr, phantom, "\cong"] & N(b,d) \arrow[d, "\upsilon"] \\
\mathcal{B}(c\otimes \overline{d}, i) \arrow[r, "\mathcal{B}(g\otimes 1,1)"'] & \mathcal{B}(b\otimes \overline{d}, i)
\end{tikzcd}
\qquad
\begin{tikzcd}
N(b,c) \arrow[r, "N(1,h)"] \arrow[d, "\upsilon"'] \arrow[dr, phantom, "\cong"] & N(b,d) \arrow[d, "\upsilon"] \\
\mathcal{B}(b\otimes \overline{c}, i) \arrow[r, "\mathcal{B}(1\otimes \overline{h},1)"'] & \mathcal{B}(b\otimes \overline{d}, i)
\end{tikzcd}$$

and chasing $h : c \to d$ in the first, and $g : b \to d$ in the second. Finally we ask that transposition plays well with the conjugation and dagger so that $\overline{\upsilon(f)} = \upsilon(f^\dagger)$.

Definition 6.5. Let $\mathcal{B}$ and $\mathcal{C}$ be monoidal $\mathcal{V}$-bicategories equipped with nuclear 2-ideals. A monoidal $\mathcal{V}$-pseudofunctor $F : \mathcal{B} \to \mathcal{C}$ is **nuclear** if it preserves the nuclear ideal.

7. Local Nuclear and Trace Ideals: Horizontal Categorification

Thus far we have worked on the vertical categorification of nuclear and trace ideals: we have upgraded sets to categories, monoidal categories to monoidal bicategories, coends to pseudocoends etc. This allows us to discuss the duality of 0-cells in $\mathcal{B}$, but not to speak about the duality of 1-cells. In this section we develop the horizontal categorification of nuclear and trace ideals to remedy this.

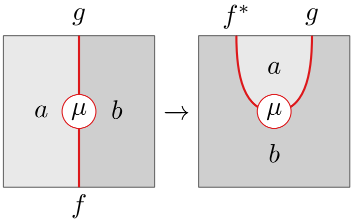
Definition 7.1. Let $\mathcal{B}$ be a monoidal $*$-bicategory. A **local** (left) **nuclear ideal** for $\mathcal{B}$ consists of an assignment of a sub-profunctor,

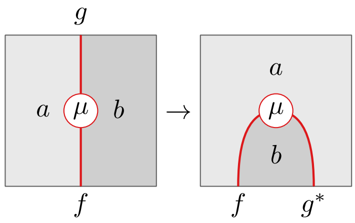
$$N(a,b)(-,-) \subseteq \mathcal{B}(a,b)(-,-) : \mathcal{B}(a,b) \nrightarrow \mathcal{B}(a,b),$$

for each pair of 0-cells (a,b). These profunctors come equipped with transformations, natural in f and g,

$$N(a,b)(f,g) \to \mathcal{B}(b,b)(1_b, gf^*), \qquad N(a,b)(f,g) \to \mathcal{B}(a,a)(g^*f, 1_a) \tag{7}$$

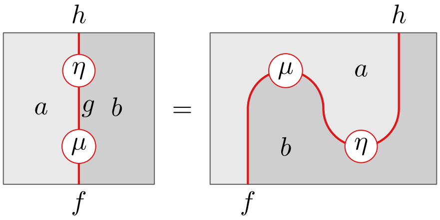
which we can picture acting as:





such that the snake equations hold:

$$\begin{tikzcd}[column sep=small]
\int^g N(a,b)(f,g) \boxtimes N(a,b)(g,h) \arrow[rr, "\int \upsilon' \times \upsilon"] \arrow[dr, "\circ"'] & & \int^g N(a,a)(g^*f, 1_a) \boxtimes N(b,b)(1_b, hg^*) \arrow[dl, "\circ"] \\
& N(a,b)(f,h) &
\end{tikzcd}$$



We ask that the collection of profunctors is stable under the action of the $(-)^\dagger$ and the conjugation $\overline{(-)}$ in the obvious way. We also ask that they are closed under the tensor product so there are transformations for every a, b, c, d,

$$N(a,b)(f,g) \boxtimes N(c,d)(h,k) \to N(a \otimes c, b \otimes d)(f \otimes h, g \otimes k)$$

natural in f, g, h, k. These transformations should be compatible in the obvious way with the transposition operations (7). Local *right* nuclear ideals are defined analogously, capturing the other snake equation.

Example 7.2. The delooping of a monoidal category with a nuclear ideal has a local nuclear ideal as a bicategory.

Example 7.3. Any bicategory with duals for 1-cells has a local nuclear ideal where $N(a,b)(-,-) = \mathcal{B}(a,b)(-,-)$ for every (a,b). This includes Prof, Cob and 2FHilb.

Definition 7.4. Let $\mathcal{B}$ be a $\mathcal{V}$-bicategory. A **local** (right) **trace ideal** for $\mathcal{B}$ consists of a sub-profunctor,

$$T(a,b)(-,-) \subseteq \mathcal{B}(a,b)(-,-) : \mathcal{B}(a,b) \nrightarrow \mathcal{B}(a,b),$$

for each pair of 0-cells a, b, and a morphism,

$$\int^f T(a,b)(f,f) \to \mathcal{B}(a,a)(1_a, 1_a).$$

Like for a local nuclear ideal, the family of profunctors should be closed under $(-)^\dagger$, $\overline{(-)}$ and the tensor product with $\mathrm{tr}(\alpha \otimes \beta) = \mathrm{tr}(\alpha) \otimes \mathrm{tr}(\beta)$.

Proposition 7.5. *Any bicategory with a local nuclear ideal has a local trace ideal.*

Proof. Define $T(a,b)(-,-) := \int^g N(a,b)(-,g) \boxtimes N(a,b)(g,-)$ with trace given by,

$$\begin{aligned}\int^f T(a,b)(f,f) &= \int^{fg} N(a,b)(f,g) \boxtimes N(a,b)(g,f) \\ &\to \int^{fg} \mathcal{B}(a,a)(1_a, gf^*) \boxtimes \mathcal{B}(a,a)(gf^*, 1_a) \xrightarrow{\circ} \mathcal{B}(a,a)(1_a, 1_a)\end{aligned}$$

□

Example 7.6. The delooping of a monoidal category with a local trace ideal also has a local trace ideal as a bicategory.

Example 7.7. Any bicategory with duals for 1-cells has a local trace ideal induced by its local nuclear ideal from Example 7.3.

8. EXTENDED CFTS

Now we work towards defining 2-extended CFTs as nuclear pseudofunctors.

$$\mathsf{Cob}_{\mathrm{conf}} \to \mathsf{2Hilb}$$

8.1. **Conformal Cobordisms.** Let us start with the domain bicategory.

Definition 8.2. $\mathsf{Cob}_{\mathrm{conf}}$ is the monoidal bicategory given by the following data.

1. The 0-cells are 0-dimensional manifolds M.
2. The 1-cells $M_1 \to M_2$ are one of:
   - (a) a diffeomorphism $M_1 \to M_2$,
   - (b) a 1-dimensional cobordism, that is, a 1-dimensional manifold Σ and a diffeomorphism $\partial\Sigma \to M_1 \sqcup M_2$.

   The composition of diffeomorphisms is straightforward and composition of cobordisms is given by gluing which we write $\Sigma_2 \cup_{M_2} \Sigma_1$. The composition of a cobordism and diffeomorphism is simply given by reinterpreting the boundary.
3. The 2-cells between two cobordisms $\Sigma_1, \Sigma_2 : M_1 \to M_2$ are one of:
   - (a) a diffeomorphism $\Sigma_1 \to \Sigma_2$,
   - (b) a 2-dimensional conformal cobordism possibly with cusps, that is, a 2-dimensional manifold S with conformal structure in the interior and a diffeomorphism $\partial S \to \Sigma_1 \cup_{M_1 \cup M_2} \Sigma_2$. Two cobordisms S and S' are considered the same if there is a conformal diffeomorphism $S \to S'$.

   The 2-cells between a diffeomorphism $\phi : M_1 \to M_2$ and a cobordism $\Sigma : M_1 \to M_2$ are given by a conformal cobordism S whose boundary is given by Σ upon gluing its boundaries together along ϕ. Composition of conformal cobordisms is given by conformal gluing, see [15].

$\mathsf{Cob}_{\mathrm{conf}}$ is monoidal under the disjoint union of manifolds. It becomes a $*$-bicategory when equipped with a $(-)^\dagger$ which acts to reverse the inputs and outputs of a cobordism: $\Sigma : M_1 \to M_2$ is sent to $\Sigma^\dagger : M_2 \to M_1$. Its behaviour on diffeomorphisms and 2-cells is obvious. The conjugation $\overline{(-)}$ is the identity on 1-cells and 2-cells.

Given a cobordism 1-cell $\Sigma : M_1 \to M_2$ we can freely reinterpret it as a cobordism $\varnothing \to M_1 \sqcup M_2$. On the other hand, the collection of diffeomorphisms $\varnothing \to M_1 \sqcup M_2$ is empty (besides the special case where the codomain is the empty manifold) and so we cannot transform diffeomorphisms $M_1 \to M_2$ in the same way. This means that only the cobordisms permit a transpose and in fact form a nuclear 2-ideal.

Proposition 8.3. *The bicategory* $\mathsf{Cob}_{\mathrm{conf}}$ *has a nuclear 2-ideal given by the cobordisms and the 2-cells on them.*

The following is then also immediate.

Proposition 8.4. *The bicategory* $\mathsf{Cob}_{\mathrm{conf}}$ *has a trace 2-ideal given by the cobordisms and the 2-cells on them.*

Similarly, any conformal cobordism 2-cell $S : \Sigma_1 \to \Sigma_2 : M_1 \to M_2$ can be reinterpreted as a conformal cobordism $1_{M_1} \to \Sigma_2 \cup_{M_2} \Sigma_1 : M_1 \to M_1$ while the diffeomorphisms do not support such a transpose.

Proposition 8.5. *The bicategory* $\mathsf{Cob}_{\mathrm{conf}}$ *has a local nuclear ideal and a local trace ideal given by the conformal cobordism 2-cells.*

8.6. **2-Hilbert Spaces and Measured Categories.** Now we turn our attention to the codomain of our 2-extended CFT. We would like to define a notion of an infinite dimensional 2-Hilbert space in which to value the CFT. There is now a well-established definition of a finite dimensional 2-Hilbert space due to [3, 4].

Definition 8.7. A **H^*-category $\mathcal{H}$** is a category enriched in FHilb together with antilinear maps $* : \mathcal{H}(a,b) \to \mathcal{H}(b,a)$ such that $f^{**} = f$, $(fg)^* = g^* f^*$, $\langle fg, h\rangle = \langle g, f^* h\rangle$ and $\langle fg, h\rangle = \langle f, hg^*\rangle$.

Definition 8.8. A **finite dimensional 2-Hilbert space** is a Cauchy complete H^*-category $\mathcal{H}$, This means that every left adjoint profunctor $\mathcal{A} \nrightarrow \mathcal{H}$ for every FHilb-category $\mathcal{A}$ is representable.

The notion of Cauchy completion in terms of profunctors dates back to the work of Lawvere [19] and Borceux and Dejean [6]. Finite dimensional 2-Hilbert spaces were originally defined as abelian H^*-categories [3] but Cauchy completeness of a 2-Hilbert space was shown to be equivalent to abeliancy in [4].

Definition 8.9. The 2-category $\mathsf{2FHilb}$ of finite dimensional 2-Hilbert spaces has 0-cells given by the finite dimensional 2-Hilbert spaces, 1-cells given by the linear $*$-functors (so that $F(f^*) = (Ff)^*$) and 2-cells given by the natural transformations.

While the notion of a finite dimensional 2-Hilbert space might seem rather opaque, there is another characterisation of the bicategory $\mathsf{2FHilb}$ in terms of the 2-vector spaces of Kapranov and Voevodsky [17].

Definition 8.10. The bicategory $\mathsf{2Mat}$ of KV 2-Hilbert spaces has 0-cells given by the natural numbers $n \in \mathbb{N}$, 1-cells $H : m \to n$ given by $m \times n$ matrices of finite dimensional Hilbert spaces H_{ij},

$$H = \begin{pmatrix} H_{11} & \dots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{m1} & \dots & H_{mn} \end{pmatrix}$$

and 2-cells $f : H \Rightarrow K$ given by matrices of linear maps $f_{ij} : H_{ij} \to K_{ij}$.

$$f = \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \dots & f_{mn} \end{pmatrix} : \begin{pmatrix} H_{11} & \dots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{m1} & \dots & H_{mn} \end{pmatrix} \to \begin{pmatrix} K_{11} & \dots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{m1} & \dots & K_{mn} \end{pmatrix}$$

The bicategory $\mathsf{2Mat}$ is equivalent to $\mathsf{2FHilb}$, mirroring the equivalence between the category FHilb of finite dimensional Hilbert spaces and Mat of natural numbers and matrices. Much like for normal finite dimensional Hilbert spaces, each finite dimensional 2-Hilbert space has a finite basis and we can identify it with the cardinality of that basis, that is, the dimension of the 2-Hilbert space [3, 16]. In particular, a finite dimensional 2-Hilbert space $\mathcal{H}$ is equivalent to the category FHilb^n for some natural number n [3].

The bicategories $\mathsf{2FHilb}$ and $\mathsf{2Mat}$ can be equipped with two different tensor products, the former is known as the Deligne tensor and is a categorification of the tensor product of Hilbert spaces. Given two finite dimensional 2-Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ their tensor $\mathcal{H} \otimes \mathcal{K}$ is given by the Cauchy completion of their enriched tensor product as FHilb-categories. This corresponds to multiplying their dimensions, $\mathsf{FHilb}^n \otimes \mathsf{FHilb}^m \cong \mathsf{FHilb}^{nm}$, so the equivalent tensor of $\mathsf{2Mat}$ is given on objects by multiplication of natural numbers. The second tensor is a categorification of the direct sum. On $\mathsf{2Mat}$ it acts on objects by addition, $n + m$. This tensor product has been considered in [8].

A notion of infinite dimensional 2-Hilbert space is less well-established. In [24] the CFTs are valued in von Neumann algebras and bimodules but given some of the complexities of that bicategory, we will instead work with the measurable categories of Yetter [25, 2]. Such categories should not be seen as a definitive definition of an infinite dimensional 2-Hilbert space, in particular they are basis-dependant being more rightfully seen as a categorification of Mat thereby giving an infinite dimensional version of $\mathsf{2Mat}$. One benefit is that direct calculation can be more tractable, but it comes at the cost of lacking a more morally correct definition.

Since all finite dimensional 2-Hilbert spaces $\mathcal{H}$ are equivalent to a category of the form FHilb^n for some natural number n, this suggests a basis-dependent definition in infinite dimensions by replacing FHilb with Hilb and n with a set X of possibly infinite cardinality. The objects of FHilb^n consist of lists $(H_1, \dots, H_n)$ of finite dimensional Hilbert spaces, which we can think of as a bundle $n \to \mathsf{FHilb}$. When n is promoted to a set X of infinite cardinality, particularly when it is uncountable, we would run into problems of convergence, for instance with direct sums indexed by X. Thankfully the theory of measurable fields of Hilbert spaces and of direct integrals allows us to bypass these problems. For this we require our sets X to be measurable spaces.

Definition 8.11. A **measurable space** (X, Σ) is a set X equipped with a σ-algebra Σ of subsets of X. We will often leave the σ-algebra implicit.

In fact we will restrict ourselves further to the standard Borel spaces.

Definition 8.12. A measurable space (X, Σ) is **standard Borel** if X can be endowed with the structure of a separable, complete metric space, such that Σ is the Borel σ-algebra on X.

The following is a standard result.

Proposition 8.13. *(X, Σ) is a standard measure space if and only if it is isomorphic to one of the following: a finite set X with its σ-algebra of all subsets; a countable set X with its σ-algebra of all subsets; or an uncountable set X with its Borel σ-algebra. In the latter case we can take $X = \mathbb{R}$.*

Definition 8.14. A **measure space** (X, Σ, μ) is a measurable space (X, Σ) equipped with a σ-finite measure $\mu : \Sigma \to \mathbb{R} \cup \{+\infty\}$.

Remark. For the avoidance of any doubt, note that all of our measures will be σ-finite so we include this in the definition of a measure space.

Definition 8.15. Let X be a standard Borel space. A **measurable field of Hilbert spaces** (H, M_H) over X consists of an assignment of a Hilbert space H_x for each $x \in X$ together with a subspace $M_H \subseteq \prod_x H_x$ of **measurable sections** such that the following properties hold,

- for any measurable section $m \in M_H$, the function $x \mapsto \|m(x)\|_{H_x}$ is measurable,
- for each $n \in \prod_x H_x$, if $x \mapsto \langle n(x) | m(x) \rangle$ is measurable for every $m \in M_H$ then $n \in M_H$,
- there is a sequence $m_i \in M_H$ such that $\{m_i(x)\}_i$ is dense in H_x for each $x \in X$.

Definition 8.16. Let H and K be measurable fields of Hilbert spaces over X. A **measurable field of bounded linear maps** $f : H \to K$ consists of a bounded linear map $f_x : H_x \to K_x$ for each $x \in X$ such that for each $m \in M_H$, we have $f(m) \in M_K$ where $f(m)_x := f_x(m(x))$.

Definition 8.17. Let (X, μ) be a measure space with X standard Borel. A measurable field of bounded linear maps $f : H \to K$ over X is **essentially bounded** if $x \mapsto \|f_x\|_x$ is in $L^\infty(X, \mu)$ where $\|-\|_x$ is the operator norm on $B(H_x, K_x)$.

For each measure space (X, μ) there is a category which we denote $\mathsf{Hilb}^{(X,\mu)}$ whose objects are the measurable fields of Hilbert spaces over X and whose morphisms are the essentially bounded measurable fields of bounded linear maps over X.

Example 8.18. When X is a finite set and μ is the counting measure we can identify $\mathsf{Hilb}^{(X,\mu)}$ with Hilb^n where $n = |X|$.

To introduce the functors between categories of the form $\mathsf{Hilb}^{(X,\mu)}$ we need the notion of a direct integral.

Definition 8.19. Let H be a measurable field of Hilbert spaces over a measure space (X, μ). The **direct integral** $\int_X^\oplus H_x \,\mathrm{d}\mu(x)$ is the Hilbert space of μ-a.e. equivalence classes of measurable L^2-sections of H. Explicitly it consists of measurable sections $m \in M_H$ such that $\int_X \|m_x\|^2 \,\mathrm{d}\mu(x) < \infty$ modulo μ-a.e. equivalence. Such an equivalence class is denoted $\int_X^\oplus m_x \,\mathrm{d}\mu(x)$.

Given an essentially bounded field of bounded linear maps $f : H \to K$ it is possible to take the direct integral to get the following linear map which acts pointwise on sections:

$$\int_X^\oplus f_x \,\mathrm{d}\mu(x) : \int_X^\oplus H_x \,\mathrm{d}\mu(x) \to \int_X^\oplus K_x \,\mathrm{d}\mu(x) :: \int_X^\oplus m_x \,\mathrm{d}\mu(x) \mapsto \int_X^\oplus f_x(m_x) \,\mathrm{d}\mu(x) \tag{8}$$

In the following definition of the matrix functors between categories of the form $\mathsf{Hilb}^{(X,\mu)}$ it will be useful to recall the notion of the **pushforward** of a measure μ on a measurable space (X, Σ_X) along a measurable function $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$. This pushforward is given by $(f_*\mu)(B) := \mu(f^{-1}B)$ for each $B \in \Sigma_Y$. Two useful examples are in order. In the case that we have the projection $\pi_X : Y \times X \to X$, the pushforward $\pi_{X*}\mu$ is simply given by $\mu(Y \times A)$ for each $A \in \Sigma_X$, and symmetrically for the projection π_Y. We will call such pushforwards the **projection** of μ onto either X or Y. In the case of the diagonal map $\Delta : X \to X \times X$, the pushforward is given by $(\Delta_*\mu)(A) = \mu(\Delta^{-1}A)$ and thus measures under μ just the diagonal part of a measurable subset $A \subseteq X \times X$. Note that $((\pi_X\Delta)_*\mu)(A) = \mu(A)$ so that the projection of $\Delta_*\mu$ onto X is absolutely continuous with respect to μ_X, that is $\pi_{X*}(\Delta_*\mu) \ll \mu_X$.

We also recall the notion of a conditional measure distribution.

Definition 8.20. Let $p : (X, \Sigma_X) \to (Y, \Sigma_Y)$ be a measurable function. A family $(\mu_y)_{y \in Y}$ of measures on X is a **conditional measure distribution** with respect to p if,

1. $\mu_y(X \backslash p^{-1}(y)) = 0$ for each $y \in Y$,
2. $y \mapsto \mu_y(A)$ is measurable for each $A \in \Sigma_X$,
3. the family is uniformly σ-finite so that there is a family $(A_n)_{n \in \mathbb{N}}$ of measurable sets $A_n \in \Sigma_X$ that covers X, ($\bigcup_n A_n = X$) with $\mu_y(A_n) < \infty$ for each $y \in Y$.

In that case that X is equipped with a measure μ and Y is equipped with a measure ν one can ask whether it is possible to decompose μ into a conditional measure distribution $(\mu_y)_{y \in Y}$ with respect to p such that,

$$\int_Y \mu_y(A) \,\mathrm{d}\nu(y) = \mu(A), \qquad \text{for each } A \in \Sigma_X.$$

This is known as a **disintegration** of μ with respect to (p, ν) and the following theorem provides the necessary and sufficient conditions for such a disintegration to exist.

Theorem 8.21 ([14])**.** *Let $p : (X, \Sigma_X) \to (Y, \Sigma_Y)$ be a measurable function and ν be a measure on Y. A measure μ on (X, Σ_X) has a disintegration with respect to (p, ν) if and only if $p_*\mu$ is absolutely continuous with respect to ν. In this case the disintegration is ν-almost everywhere unique.*

Suppose now that we are given a measure ν on $Y \times X$ whose projections onto X and Y are absolutely continuous with respect to μ_X and μ_Y. Then by Theorem 8.21, ν possesses disintegrations,

$$\nu = \int_X \nu_x \,\mathrm{d}\mu_X(x) \qquad \text{and} \qquad \nu = \int_Y \nu_y \,\mathrm{d}\mu_Y(y)\,,$$

with $(\nu_x)_{x \in X}$ a conditional measure distribution with respect to π_X and $(\nu_y)_{y \in Y}$ a conditional measure distribution with respect to π_Y. Such a distribution was known as an X**-fibred** (respectively Y-fibred) **measure distribution on** $Y \times X$ in [2].

Definition 8.22. Let $F_{y,x}$ be a measurable field of Hilbert spaces on $Y \times X$ and ν be a measure on $Y \times X$ such that the projections $\pi_{X*}\nu$ and $\pi_{Y*}\nu$ are absolutely continuous with respect to μ_X and μ_Y respectively, that is,

$$\pi_{X*}\nu \ll \mu_X \qquad \text{and} \qquad \pi_{Y*}\nu \ll \mu_Y .$$

The **matrix functor** $(F,\nu) : \mathsf{Hilb}^{(X,\mu_X)} \to \mathsf{Hilb}^{(Y,\mu_Y)}$ induced by $F_{y,x}$ and ν acts to send a measurable field of Hilbert spaces H_x over X to the measurable field over Y,

$$(FH)_y := \int_X^{\oplus} F_{y,x} \otimes H_x \, \mathrm{d}\nu_y \,,$$

where $(\nu_y)_{y\in Y}$ is the Y-fibred measure distribution given by disintegration of ν with respect to μ_Y. On the essentially bounded fields of bounded linear maps, it acts in the obvious pointwise way on sections analogously to (8). We identify two matrix functors (F,ν) and (G,ν) if the fields F and G are ν-almost everywhere equal.

Given two matrix functors $\mathsf{Hilb}^{(X,\mu_X)} \xrightarrow{(F,\nu)} \mathsf{Hilb}^{(Y,\mu_Y)} \xrightarrow{(G,\xi)} \mathsf{Hilb}^{(Z,\mu_Z)}$ we can compose them in essentially the same way as outlined in [25, 2]. To get the measure $\xi\nu$ on $Z \times X$, we disintegrate ν with respect to μ_Y to yield the Y-fibred measure distribution ν_y on $Y \times X$ and ξ with respect to μ_Z to yield the Z-fibred measure distribution ξ_z on $Z \times Y$. Then we integrate,

$$(\xi\nu)_z := \int_Y \nu_y \, \mathrm{d}\xi_z \,,$$

to yield a Z-fibred measure distribution on $Z \times X$ and finally integrate with respect to the measure ξ on Z,

$$\xi\nu := \int_Z \delta_z \otimes (\xi\nu)_z \, \mathrm{d}\xi \,.$$

The composition of the fields works similarly, by direct integral over the disintegration of the measures, see [25, 2] for more of the details.

Definition 8.23. Let $(F,\nu^F), (G,\nu^G) : (X,\mu_X) \to (Y,\mu_Y)$ be two 1-cells and $\alpha : F \to G$ be a $(\nu^F + \nu^G)$-essentially bounded measurable field of bounded linear maps. By the results of [25], α induces a **matrix natural transformation** $\alpha : (F,\nu^F) \to (G,\nu^G)$ between the matrix functors $(F,\nu_F), (G,\nu_G) : \mathsf{Hilb}^{(X,\mu_X)} \to \mathsf{Hilb}^{(Y,\mu_Y)}$ with the following components.

$$\begin{aligned}(\alpha_H)_y : \int_X^{\oplus} F_{y,x} \otimes H_x \, \mathrm{d}\nu_y^F &\to \int_X^{\oplus} G_{y,x} \otimes H_x \, \mathrm{d}\nu_y^G \\ :: \int_X^{\oplus} m_{y,x} \otimes n_x \, \mathrm{d}\nu_y^F &\mapsto \int_X^{\oplus} \sqrt{\frac{\mathrm{d}\nu_y^F}{\mathrm{d}\nu_y^G}} \alpha_{y,x}(m_{y,x}) \otimes n_x \, \mathrm{d}\nu_y^G\end{aligned}$$

We identify two matrix natural transformations that are $\sqrt{\nu^F\nu^G}$-almost everywhere equivalent in the sense described in [2].

Definition 8.24. We write Meas for the bicategory of categories of the form $\mathsf{Hilb}^{(X,\mu_X)}$, matrix functors and bounded matrix transformations. It is monoidal under the product of measure spaces, $\mathsf{Hilb}^{(X,\mu_X)} \times \mathsf{Hilb}^{(Y,\mu_Y)} = \mathsf{Hilb}^{(X\times Y,\mu_X\times\mu_Y)}$. It becomes a $*$-bicategory with $(-)^\dagger$ given by reinterpreting a matrix functor $(F,\nu) : (X,\mu_X) \to (Y,\mu_Y)$ as one $(F,\nu) : (Y,\mu_Y) \to (X,\mu_X)$, noting that this is possible because the projections of ν are absolutely continuous with respect to both μ_X and μ_Y. The conjugation sends (F,ν) to $(\overline{F},\nu)$ where $\overline{F}$ is the measurable field of Hilbert spaces over $Y \times X$ with the Hilbert space over each fibre given by the complex conjugate of that of F, $(\overline{F})_{y,x} = \overline{F}_{y,x}$.

Proposition 8.25. *When (X, μ) is discrete (so X is a finite set or $\mathbb{N}$ and μ is the counting measure)* $\mathsf{Hilb}^{(X,\mu_X)}$ *is dualisable.*

Proof. The dual 0-cell is $\mathsf{Hilb}^{(X,\mu_X)}$ itself. The cap $\eta : X \times X \to 1$ is given by the diagonal measurable field of Hilbert spaces over $X \times X$, explicitly $\eta_{x,x'} = \mathbb{C}$ when $x = x'$ and 0 otherwise, together with the diagonal measure on $X \times X$ given by the pushforward $\Delta_*\mu$. Similarly, the cup $\varepsilon : 1 \to X \times X$ is given by the same diagonal measurable field of Hilbert spaces and diagonal measure on $X \times X$. In both cases, absolute continutity of the projections is straightforward to check and amounts to the requirement that $\Delta_*\mu \ll \mu \times \mu$. Any measure ν on $X \times X$ is absolutely continuous with respect to the product $\mu \times \mu$ because $(\mu \times \mu)(A) = 0$ implies that $A = \varnothing$ and clearly $\nu(\varnothing) = 0$. $\square$

One of the main points to notice in the previous proof is that for discrete spaces with μ the counting measure, *any* other measure ν on $Y \times X$ is absolutely continuous with respect to $\mu_Y \times \mu_X$. This allows to view any 1-cell $(X, \mu_X) \to (Y, \mu_Y)$ between discrete spaces equally as a 1-cell $1 \to (Y \times X, \mu_Y \times \mu_X)$ or as a 1-cell $(Y \times X, \mu_Y \times \mu_X) \to 1$ setting up isomorphisms,

$$\mathsf{Meas}((X, \mu_X), (Y, \mu_Y)) \cong \mathsf{Meas}(1, (Y \times X, \mu_Y \times \mu_X)) \cong \mathsf{Meas}((Y \times X, \mu_Y \times \mu_X), 1). \tag{9}$$

On the other hand, in the continuous infinite case, $X = \mathbb{R}$ equipped with the Lebesgue measure, $\mathsf{Hilb}^{(X,\mu)}$ is not dualisable. This is because not every measure on $Y \times X$ is absolutely continuous with respect to the product measure. For instance, consider the diagonal measure $\Delta_*\mu$. The product measure is zero on the diagonal:

$$(\mu \times \mu)(\Delta\mathbb{R}) = \int_{\mathbb{R}} \mu(\{*\}) \, \mathrm{d}\mu = 0,$$

while $(\Delta_*\mu)(\Delta\mathbb{R}) = \mu(\Delta^{-1}\Delta\mathbb{R}) = \mu(\mathbb{R}) > 0$ so the diagonal measure is not absolutely continuous with respect to the product measure. The prevents the isomorphisms (9) from holding in the continuous case and motivates the following definition.

Definition 8.26. A 1-cell $(F, \nu) : (X, \mu_X) \to (Y, \mu_Y)$ is **Hilbert-Schmidt** when $\nu \ll \mu_Y \times \mu_X$.

Proposition 8.27. *The bicategory* Meas *has a nuclear 2-ideal given by the Hilbert-Schmidt 1-cells together with the 2-cells on them.*

As a consequence we can make the following definition.

Definition 8.28. A 1-cell $(F, \nu) : (X, \mu_X) \to (Y, \mu_Y)$ is **trace class** if it is isomorphic to the composite of two Hilbert-Schmidt 1-cells.

The following proposition is then immediate.

Proposition 8.29. *The bicategory* Meas *has a trace 2-ideal given by the trace class 1-cells together with the 2-cells on them.*

Remark. It is worth commenting on another potential definition for traces of 1-cells in Meas. Given $(F, \nu) : (X, \mu_X) \to (X, \mu_X)$, one might attempt to take the pullback of ν along the diagonal and define,

$$\mathrm{tr}(F, \nu) = \int_X^{\oplus} F_{x,x} \, \mathrm{d}(\nu\Delta) \, .$$

While for discrete X, this coincides with definition given by nuclearity, it does not coincide for continuous X. For any measure $\nu \ll \mu \times \mu$ and any measurable $A \subseteq X$, $(\mu \times \mu)(\Delta A) = 0$ so that $\nu(\Delta A) = 0$. Thus $\nu\Delta$ is the zero measure and $\mathrm{tr}(F, \nu) = 0$.

Now let us turn our attention to the 2-cells that permit a transpose.

Definition 8.30. A 2-cell $\alpha : (F, \nu^F) \to (G, \nu^G)$ is **Hilbert-Schmidt** when

$$\int_{X \times Y} \|\alpha_{x,y}\|^2_{\mathrm{HS}} \, \mathrm{d}\sqrt{\nu^F \nu^G} < \infty,$$

where $\|-\|_{\mathrm{HS}}$ is the usual Hilbert-Schmidt norm.

Proposition 8.31. *The bicategory* Meas *has a local nuclear ideal given by the Hilbert-Schmidt 2-cells.*

Proposition 8.32. *There is an equivalence of bicategories between* Meas *and the bicateory* avN *of abelian von Neumann algebras and correspondences.*

Proof. (Sketch) Given a measure space (X, μ) we send it to the abelian von Neumann algebra $L^\infty(X, \mu)$. A correspondence $L^\infty(X, \mu_X) \to L^\infty(Y, \mu_Y)$ is always given by a measurable field of Hilbert spaces over the product $Y \times X$ together with a measure on $Y \times X$ whose projections are absolutely continuous with respect to those on X and Y, see [10]. ☐

8.33. **Extended CFTs.** Finally we are in a position to give the definition of a once extended CFT.

Definition 8.34. A generalised CFT is a monoidal nuclear functor $Z : \mathsf{Cob}_{\mathrm{conf}} \to \mathcal{B}$ where $\mathcal{B}$ is a monoidal bicategory with local nuclear ideals and a nuclear 2-ideal. This means Z should preserve both the local nuclear ideals and the nuclear 2-ideal.

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