Monoidal structures for hypergraphs and transpositionality (Abstract)

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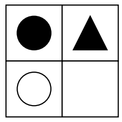
Analogical reasoning operates by transposing some pattern along another one. For such transposition to be possible, the ambient structure must be somewhat alike a cartesian product. For instance, let us consider a most basic case of Raven matrix:



In this example, one is presented with a subobject of the cartesian product

$$\{\bigcirc, \triangle\} \times \{black,\ white\}$$

and is expected to complete it; this is an instance of an analogy problem

$$\textit{black circle : black triangle :: white circle : ?}$$

and it is solved by transporting the relation between the element of a row along a transverse one, as if the cartesian product was endowed with an implicit ternary operation satisfying

$$\textit{white triangle} = \textit{black triangle} \times_{\textit{black circle}} \textit{white circle}$$

Formalizations of analogy calculus for binary relations have been proposed, for instance in [7], [1] or [8]. In this work, we will present a new approach, not restricted to binary relations, using algebraic structures on *hypergraphs*. Hypergraphs have become a recurring topic in several areas of applied mathematics, from hypergraph neural networks used to capture higher order incidences between data ([5]) to multi-omic integration. They are a generalization of binary, undirected graphs where links may contain more than two nodes, though the extends of this generalization depends on definitions. Ours will be the most general: we call *hypergraph* a span in **Set**

$$S \xleftarrow{\sigma} T \xrightarrow{\lambda} L$$

where elements of T will be called *witnesses*, elements of S *nodes*, and elements of L *links*, and one will write $t : x \in l$ to say that t witnesses that a node x appears in the link l. This definition allows, on one hand, a same node to be several times in a same link, and on the other hand, several links to contain the same nodes with the same number of occurrences. In other words, a hypergraph is the same thing as a presheaf on the walking cospan, and in the vein of [6], we may take as a convenient category of hypergraphs the presheaf topos

$$\mathbf{Hyp} = \mathbf{Set}^{\wedge}$$

In contrast with more restricted definition as [2], [3] or [4], this endows **Hyp** with very pleasurable properties, as being complete and cocomplete, locally finitely presentable, locally cartesian closed, and allows moreover to define a dual equivalence $(-)^*$ interchanging nodes and links.

However we will be more interested in the monoidal aspects of **Hyp**. Aside its cartesian product, **Hyp** enjoys several monoidal structures generalizing some well known structures on graphs:

– the *funny tensor product* $H \square G$ where one adds copies of G-links at each H-node and conversely, producing two classes of "transverse" links;

– the *straight tensor product* $H \diamondsuit G$ related to the funny product through a De-Morgan law

$$(H \diamondsuit G)^* \simeq H^* \square G^*$$

– the *strong tensor product* which is a combination of the funny and cartesian product.

We show here that each of these monoidal structures is symmetric monoidal closed, and explain how the links in the corresponding homgraphs represent notions of "2-cells" between hypergraphs morphisms. They also defines two commutative monads $(-)\square 1$ and $(-)\diamondsuit 1$ whose algebras can be seen respectively as "hypergraphs whose nodes have units" and "hypergraphs with pointed links".

Moreover, the funny tensor product is of peculiar interest in formalizing the aforementioned analogy problem, as hypergraphs of the form $H \square G$ are naturally endowed with an operation of "links transposition". Generalizing this structure, we will say that a hypergraph H has *transpositions* if for any intersection of links at a same node $t : x \in l, \quad t' : x \in l'$ and any $r : y \in l$, there is a *t-transposition of l' along r*, denoted $r \otimes_{t,t'} l'$ together with *transposed nodes and witnesses* at each $r' : y' \in l'$

$$r \otimes_{t,t'} r' : r \otimes_{t,t'} y' \in r \otimes_{t,t'} l'$$

which is to be read that "$r \otimes_{t,t'} y'$ is to y what y' is to x", though as part of a possibly larger pattern coded by l' transposed at y and in a way that is sensitive to multiple occurrences. Such a law may be enhanced with different axioms akin to those of *proportoids* introduced in [1], and combined with algebra structure for the $(-)\square 1$ monad providing unit links that behave like neutral elements for transpositions.

We discuss monadicity of such categories of transpositional hypergraphs over the category of hypergraphs, and explain why the corresponding forgetful functor is finitary. We also show that the category of transpositional graphs inherits the funny tensor product. We finally provide a few implementation of analogy calculus in this framework and discuss some relation with the notion of *commutation* in linguistic.

References

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