Double categories for adaptive quantum computation

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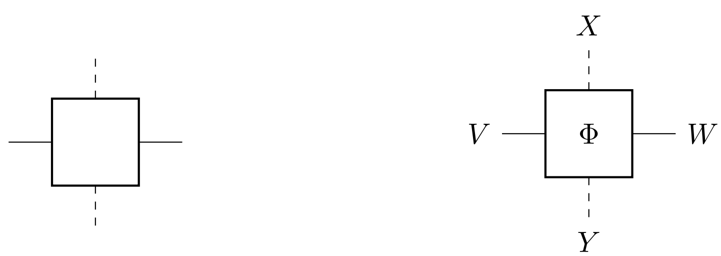
Quantum computation can be formulated in various models, each highlighting different structural and resource-theoretic aspects of quantum computational power. This paper [1] develops a unified categorical framework encompassing these models and their interrelations through the language of double categories. A port graph [2] is a directed graph with distinguished dangling edges serving as inputs and outputs, and is used to represent circuit-like diagrams. We introduce *double port graphs*, a bidirectional generalization of port graphs, to represent the quantum (horizontal) and classical (vertical) flows of information within computational architectures; see Figure 1a.



(a) (b)

Figure 1: (a) A double port graph consisting of a single node. (b) An adaptive instrument Φ with input X output Y operating on the input Hilbert space V with output Hilbert space W. The horizontal direction indicated the quantum computation, whereas the vertical direction is the classical computation.

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We describe quantum operations as *adaptive instruments*, organized into a one-object double category Inst whose horizontal and vertical directions correspond to quantum channels and stochastic maps, respectively. More precisely, an adaptive instrument is a function $\Phi : X \times Y \to \mathrm{CV}(V, W)$ with input set X and output set Y, taking values in completely positive maps $V \to W$; see Figure 1b. Within this framework, we capture prominent models of adaptive quantum computation, including the circuit model [3], the measurement-based model [4], and the magic-state model [5]. These basic models are represented as double categories of labeled port graphs, with labelings $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{Q}$, respectively. In this way, conversions between these models can be expressed as double functors:

$$\begin{array}{ccccc} & & \mathsf{DPG}_{\mathcal{Q}} & & \\ & \overset{\kappa}{\nearrow} & & \overset{\phi}{\searrow} & \\ \mathsf{DPG}_{\mathcal{C}} & & & & \mathsf{Inst} \\ & \underset{\kappa}{\searrow} & & \underset{\phi}{\nearrow} & \\ & & \mathsf{DPG}_{\mathcal{M}} & & \end{array} \tag{1}$$

To analyze computational power in a uniform way, we extend the theory of contextuality to an adaptive setting through the notion of *simplicial instruments*, which generalize simplicial distributions [6] in a double-categorical direction. Within this framework, we define Bell instruments built from local and adaptive instruments, and describe how the simplicial distribution associated with a Bell instrument and a quantum state can be represented, in analogy with the Born rule of quantum mechanics, by means of simplicial Bell scenarios. This construction yields a quantitative characterization of computational power in terms of contextual fraction. In particular, our categorical formulation captures a well-known result of [7, 8] by identifying contextual simplicial distributions as potential candidates for the implementation of Boolean functions that are not affine.

Theorem. *Let $f : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$ be a Boolean function, and let (ρ, Φ, h) be a triple consisting of a quantum state ρ, an m-Bell instrument Φ, and an affine Boolean function $h : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$ that together compute f. If the simplicial distribution associated with the pair (ρ, Φ) has non-empty support, then f must be affine.*

Our framework offers a new perspective on the interplay between adaptivity, contextuality, and computational power in quantum computational models. Double-categorical diagrams allow us to represent the operational aspects of computational models, while simplicial instruments capture the contextual behavior of the resulting distributions; in turn, these shed light on the computational power of the model under consideration. The double-categorical perspective has also proved useful in the study of quantum advantage via the classical simulation of quantum computational models using polyhedral simulators; see [9].

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Abstract

Quantum computation can be formulated through various models, each highlighting distinct structural and resource-theoretic aspects of quantum computational power. This paper develops a unified categorical framework that encompasses these models and their interrelations using the language of double categories. We introduce *double port graphs*, a bidirectional generalization of port graphs, to represent the quantum (horizontal) and classical (vertical) flows of information within computational architectures. Quantum operations are described as *adaptive instruments*, organized into a one-object double category whose horizontal and vertical directions correspond to quantum channels and stochastic maps, respectively. Within this setting, we capture prominent adaptive quantum computation models, including measurement-based and magic-state models. To analyze computational power, we extend the theory of contextuality to an adaptive setting through the notion of *simplicial instruments*, which generalize simplicial distributions to double categorical form. This construction yields a quantitative characterization of computational power in terms of contextual fraction, leading to a categorical formulation of the result that non-contextual resources can compute only affine Boolean functions. The framework thus offers a new perspective on the interplay between adaptivity, contextuality, and computational power in quantum computational models.

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1 Introduction

Quantum computation can be modeled in different ways, each emphasizing different resource requirements for computational power and efficiency. The precise connection between these resources and their computational effects is an active area of research in quantum advantage. Category theory provides a systematic language for analyzing and comparing such models, allowing structural similarities and differences to be expressed at a high level of abstraction. For example, categorical quantum mechanics describes quantum processes through string diagrams

[1, 2, 3, 4, 5]. This formalism abstracts the essential features of the Hilbert space formulation of quantum theory, offering deeper insights into the emergence of new phenomena. Fundamental constructions in quantum information processing, such as quantum teleportation, can thus be studied at varying levels of abstraction. The string diagram approach is 1-dimensional in the sense that it primarily represents Hilbert space operations—namely, quantum channels. In this work, we separate the classical and quantum directions within a double categorical framework, clarifying how these two modes of computation interact and thereby providing a more refined explanation of the principles, such as quantum contextuality, underlying quantum advantage.

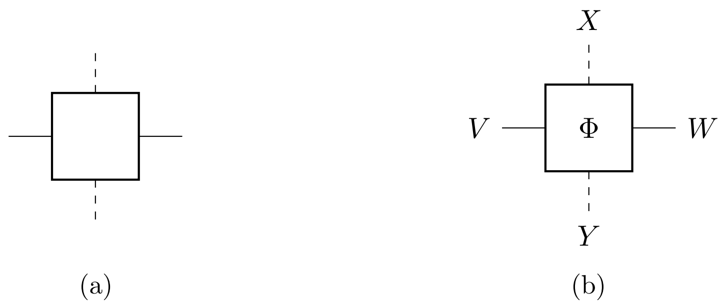

Figure 1: (a) A double port graph consisting of a single node. (b) An adaptive instrument Φ with input X output Y operating on the input Hilbert space V with output Hilbert space W. The horizontal direction indicated the quantum computation, whereas the vertical direction is the classical computation.

Our philosophy is to adopt a high-level treatment of operations represented by boxes with two kinds of composition. Such a basic component is depicted as in Figure 1a where solid horizontal wires denote qubits and dashed vertical wires denote bits. Thus, quantum information flows along solid wires in the horizontal direction, while classical information flows along dashed wires in the vertical direction. We formalize this structure as a *double port graph*. A double port graph extends the 1-dimensional notion of a port graph [6], which is a graph with distinguished dangling edges for inputs and outputs. A key property of double port graphs is that, when regarded as a 1-dimensional object (i.e., interpreting dashed wires as solid), one recovers an ordinary port graph. Treating each box as a vertex yields a directed graph, which naturally induces a temporal order on the operations, both horizontally and vertically. This temporal order has a causal nature: if an operation is controlled by the output of another, then its own outputs cannot be used as inputs to the controlling box. We assemble labeled double port graphs into a double category $\mathsf{DPG}_{\mathcal{L}}$, relative to a label set $\mathcal{L}$ (Section 2). Our notion of double category is weak in both directions of composition and is presented in the one-object setting (Section A).

Quantum operations, including unitary transformations and measurements, can be described as *instruments* [7]. An instrument is a collection of completely positive linear maps, indexed by an output set, whose sum is a channel. We adapt a more general notion of an instrument Φ, which also accepts inputs, as illustrated in Figure 1b. The input controls the instrument that will be applied at a box. A typical example is the case of adaptive quantum measurements where the measurement type, usually specified by an angle, is determined by the output of a previous measurement. Instruments are organized into a one-object double category, denoted by Inst (Section 3). This double category is essential in our framework, as it provides the semantics of the labeled boxes representing operations. Thus, our basic unit for an adaptive quantum operation is an instrument, and the double category encodes their horizontal and vertical composition. A one-object double category has an associated horizontal and vertical monoidal categories. Restricting to the horizontal direction yields the category Chan of finite-dimensional Hilbert spaces and quantum channels, while restricting to the vertical direction yields the Kleisli category Set_D of the distribution monad D [8].

Our formulation of quantum computation begins with the circuit model [9], and then extends to measurement-based quantum computation (MBQC) [10] and quantum computation with magic states (QCM) [11]. These fundamental models are represented as double categories of labeled port graphs, $\mathsf{DPG}_{\mathcal{C}}$, $\mathsf{DPG}_{\mathcal{M}}$, and $\mathsf{DPG}_{\mathcal{Q}}$ (introduced in Sections 4, 5.1, 5.2), respectively. The label sets specify the computational primitives of each model, from which all other allowed operations can be constructed by horizontal and vertical composition of labeled boxes. It is possible to translate a computation in one model into an equivalent computation in another, where equivalence means that the associated instruments compose to the same overall operation. Formally, these conversions, together with the instrument assignment, are captured by double functors, giving the commutative diagram

$$\begin{array}{ccccc} & & \mathsf{DPG}_{\mathcal{Q}} & & \\ & \overset{\kappa}{\nearrow} & & \overset{\phi}{\searrow} & \\ \mathsf{DPG}_{\mathcal{C}} & & & & \mathsf{QBit} \\ & \underset{\kappa}{\searrow} & & \underset{\phi}{\nearrow} & \\ & & \mathsf{DPG}_{\mathcal{M}} & & \end{array} \tag{1}$$

Here, QBit denotes the double subcategory of instruments whose horizontal monoidal category is the full subcategory of QChan with objects qubit Hilbert spaces and the vertical one the Kleisli category Bool_D of Boolean maps. We write κ for the model conversion and ϕ for the instrument assignment, with subscripts indicating the model, e.g., $\kappa_{\mathcal{C},\mathcal{Q}}$ and $\phi_{\mathcal{C}}$. The model conversion is implemented by a pasting operation on labeled double port graphs and usually implemented by a choice of special *gadgets* that provide an implementation of a instrument label in one model using the labels of the other model. For each of these categories, the image of ϕ has its horizontal category landing in QChan. In practice, quantum computation seeks to approximately implement any unitary map, a notion known as *quantum universality.* The circuit model is inherently designed for this purpose, and hence the horizontal image of $\phi_{\mathcal{C}}$ lands in a subcategory of QUni, consisting of qubit Hilbert spaces and unitary maps. By contrast, the images of MBQC and QCM include not only unitaries but also other channels. The distinguishing feature of such measurement-based models is the use of *adaptivity*, represented by dashed wires. A dashed-wire input encodes a control that can affect the operation associated with a box label. A typical example in MBQC is a measurement performed at angle $\pm\alpha$ with respect to the X-Pauli axis on the Bloch sphere. Similarly, a controlled unitary may appear as a correction to ensure deterministic implementation of a unitary map. In this way, adaptivity counteracts the probabilistic nature of quantum measurements.

At the intersection of MBQC and QCM lies an interesting universal model known as measurement-based Pauli computation (MBPC) [12, 13]. The computational power of MBQC derives from measurement angles, such as $\alpha = \pi/4$, while the power of QCM comes from preparing X-eigenstates rotated by angle $\alpha = \pi/4$. In fact, in Diagram 1, the model conversion from the circuit model to MBQC actually lands in $\mathcal{M}[\pi/4]$, where the measurement labels are restricted to the angles $\{0, \pm\pi/4\}$. In MBPC, whose label set is denoted by $\mathcal{P}$, the state preparations are inherited from QCM, while the entanglement structure follows that of MBQC, with measurements restricted to the Pauli X and Y observables (i.e., $\alpha = \pi/2$). To ensure the deterministic implementation of unitary maps, we extend the label set to $\widetilde{\mathcal{P}}$ by including correction boxes. The corresponding model conversions and instrument assignments yield the following commutative diagram:

$$
\begin{array}{ccccc}
 & & \mathsf{DPG}_{\mathcal{C}} & \xrightarrow{\quad\phi\quad} & \mathsf{QBit} \\
 & & \big\downarrow{\scriptstyle\kappa} & & \big\uparrow{\scriptstyle\phi} \\
\mathsf{DPG}_{\mathcal{P}} & \hookrightarrow & \mathsf{DPG}_{\widetilde{\mathcal{P}}} & \xrightarrow{\quad\kappa\quad} & \mathsf{DPG}_{\mathcal{Q}}
\end{array}
\qquad (2)
$$

This completes the family of computational models considered in this paper. More generally, one can define model conversions between any two computational models.

The final part of our paper (Section 6) focuses on the interaction between the quantum and classical directions by providing a quantitative measure of computational power. More precisely, we use quantum non-locality and contextuality, formulated in terms of simplicial distributions [14], to identify those quantum operations that yield a classical computational boost. A typical construction of this kind, which we refer to as the OR-gadget [15], employs a Greenberger–Horne–Zeilinger (GHZ) state together with X/Y Pauli measurements. This gadget can be represented in MBPC. All of our computational models are restricted to generators of affine Boolean maps in the vertical direction, such as the XOR gate. However, the OR-gadget enables the implementation of a Boolean function that is not affine, namely the OR gate. We provide a general framework for this phenomenon by generalizing the main result of [16]:

Theorem. *Let $f : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$ be a Boolean function, and let (ρ, Φ, h) be a triple consisting of a quantum state ρ, an m-Bell instrument Φ, and an affine Boolean function $h : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$ that together compute f. If the simplicial distribution associated with the pair (ρ, Φ) has non-empty support, then f must be affine.*

To formulate this result and its refinement that uses contextual fraction extending [17], we introduce the double category sInst of simplicial instruments, where the vertical direction admits input and output spaces represented by simplicial sets. This framework generalizes simplicial distributions and measurements from [14]. The double categorical setting allows us to make precise the notions of locality of instruments and adaptivity relations. Within this setting, we define Bell instruments consisting of such local and adaptive instruments. As a second ingredient, we describe how to represent the simplicial distribution associated with a Bell instrument and a quantum state, in analogy with the Born rule of quantum mechanics, by means of simplicial Bell scenarios. The input space is an m-dimensional sphere S^m, triangulated via an iterative join construction of two points:

$$p_\rho(\Phi) : S^m \longrightarrow D(\Upsilon^1).$$

The output space Υ^1 is the symmetric 1-simplex, representing the unoriented version of the 1-simplex. The Boolean function computed by such a simplicial distribution is obtained from the generating simplices of S^m, namely the m-simplices indexed by elements of $\mathbb{Z}_2^m$, which correspond to the vertices appearing in the join construction of the simplex. The result above identifies contextual simplicial distributions as potential candidates for implementation of Boolean functions that are not affine. Our framework thus provides a general setting for analyzing computational power in terms of non-locality and adaptivity, and can be applied to a wide range of constructions in quantum computation [18, 19]. Formulating an adaptive version of contextuality applicable to the study of adaptive quantum computation has long been an open problem. Earlier topological approaches include [20, 21]. We expect that the framework developed in this paper offers a new perspective for analyzing such models.

The paper is organized as follows. Section 2 introduces a generalization of port graphs, referred to as *double port graphs*. Section 3 presents the double category of adaptive instruments. In this section, we define the double port graph representations of both quantum and Boolean circuit models, together with the corresponding double functors to adaptive instruments. We also introduce *adaptive local instruments* as a preparation for the study of adaptive quantum computational models and their computational power. Section 5 describes the main models of adaptive computation, including the measurement-based and magic-state models. Section 6 introduces *simplicial instruments* and applies them to derive our main result (Theorem 77), which quantifies computational power in terms of the contextual fraction. Appendix A summarizes preliminaries on double categories, while Appendix B derives the standard form of MBQC within our double categorical framework.

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2 Double port graphs

In this section, we introduce *double port graphs*. Intuitively, these are special kinds of graphs equipped with both horizontal and vertical inputs and outputs, allowing composition in two distinct directions—horizontal and vertical—thus forming a double category. We also define a labeled version, in which each node is assigned a label from a specified set. This section provides the technical groundwork used to formalize various quantum computational models developed in the following sections. Throughout, by a double category we mean the strict version given in Definition 82.

2.1 Port graphs

In what follows, we will use the term *port graph*, though our notion differs from that of Spivak [6] in that the input and output labels of a vertex form an *ordered* set. Let us begin by fixing some notation. We will denote by Ord and Fin the category of linearly ordered finite sets and the category of finite sets. We will denote by $\mathbb{N}$ the set of natural numbers, including 0, that is, the set of isomorphism classes in Fin or Ord. The skeletal versions of Ord and Fin will be denoted by $\mathsf{Ord}_{\mathbb{N}}$ and $\mathsf{Fin}_{\mathbb{N}}$, respectively, and their objects will be denoted by $\underline{n} = \{1, \ldots, n\}$ for $n \geq 0$, with the convention that $\underline{0}$ is the empty set.

Definition 1. For $n, m \in \mathbb{N}$, define an (m, n) *pre-port graph* Γ to consist of the following data.

- A set X called the *vertices* of Γ.
- A pair of functions

$$\mathrm{in} : X \longrightarrow \mathrm{Ord} \qquad \mathrm{out} : X \longrightarrow \mathrm{Ord}$$

  which assign to each vertex sets of inputs and outputs, respectively. We will write

$$O := \coprod_{x \in X} \mathrm{out}(x)$$
$$I := \coprod_{x \in X} \mathrm{in}(x)$$

  (where the coproduct is taken in Fin, i.e., is the disjoint union of sets) for the sets of all inputs and outputs, respectively.

- A bijection

$$\iota : \underline{m} \amalg O \longrightarrow I \amalg \underline{n}.$$

An *isomorphism of* (m, n)*-port graphs* consists of a bijection of the vertex sets and natural bijections between the in- and out-maps, such that the induced map on sets of all inputs and sets of all outputs commutes with the ι's.

To make sense of this definition, we must define the internal flow graph.

Definition 2. Let $(X, \mathrm{out}, \mathrm{in}, \iota)$ be an (m, n) pre-port graph. The corresponding *internal flow graph* is the oriented graph with vertices $X \amalg \underline{n} \amalg \underline{m}$, set of edges $\underline{m} \amalg O$, and source and target maps given by

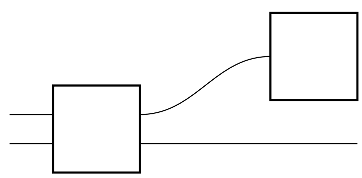
$$s(e) = \begin{cases} x & e = \mathrm{out}(x) \\ e & e \in \underline{m} \end{cases}$$

and

$$t(e) = \begin{cases} x & \iota(e) = \mathrm{in}(x) \\ \iota(e) & \iota(e) \in \underline{n}. \end{cases}$$

Definition 3. An (m, n)*-port graph* is an (m, n)-pre-port graph satisfying the condition that the internal flow graph is acyclic.

Example 4. Here is a $(2, 1)$-port graph where the vertices are depicted as boxes:



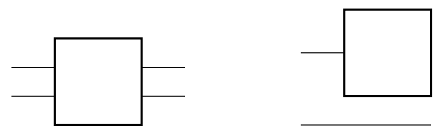
We define a concatenation operation on port graphs as follows. Let $\Gamma = (X, \mathrm{in}^X, \mathrm{out}^X, \iota^X)$ be an (m, n) port graph, and let $\Xi = (Y, \mathrm{in}^Y, \mathrm{out}^Y, \iota^Y)$ be an (n, k) port graph. We define a (m, k) pre-port graph $\Xi \circ \Gamma$ as follows.

- The vertex set of $\Xi \circ \Gamma$ is $Y \amalg X$.
- The in- and out- functions are determined by the universal property of pushout.
- The bijection ι is determined by the commutative diagram

$$\begin{array}{ccc} \underline{m} \amalg O \;\cong\; \underline{m} \amalg O^X \amalg O^Y & \longrightarrow & I^X \amalg I^Y \amalg \underline{k} \cong I \amalg \underline{k} \\ \big\downarrow & & \big\uparrow {\scriptstyle \mathrm{id} \amalg \iota^Y} \\ \underline{m} \amalg O^X \amalg \underline{n} \amalg O^Y & & \\ {\scriptstyle \iota^X \amalg \mathrm{id}} \big\downarrow & & \\ I^X \amalg \underline{n} \amalg \underline{n} \amalg O^Y & \xrightarrow[\mathrm{id} \amalg \nabla \amalg \mathrm{id}]{} & I^X \amalg \underline{n} \amalg O^Y \end{array}$$

  where ∇ is the codiagonal map on $\underline{n}$, that is, the map which acts as the identity on each copy of $\underline{n}$.

Example 5. The port graph in Example 4 is the concatenation of the following smaller port graphs



Proposition 6. *The concatenation of an (m, n)-port graph with an (n, k) port graph is an (m, k) port graph. The operation of concatenation is associative and unital, with units given by the unique (n, n) port graphs with empty vertex set. Given isomorphisms $\Gamma \cong \Gamma'$ and $\Xi \cong \Xi'$ of port graphs, there is a unique isomorphism $\Xi \circ \Gamma \cong \Xi' \circ \Gamma'$ which restricts to the original isomorphism.*

Definition 7. The *category of port graphs*, denoted by PG, has objects given by $\mathbb{N}$ and morphisms given by isomorphism classes of port graphs.

2.2 Double port graphs

Definition 8. Given U and V in Ord, the *ordinal sum* $U \oplus V$ of U and V is the linearly ordered finite set whose underlying set is $U \amalg V$, and whose order is given by

$$a < b \Leftrightarrow \begin{cases} a <_U b & a, b \in U \\ a <_V b & a, b \in V \\ a \in U, b \in V & \text{else} \end{cases}$$

This defines a monoidal structure on Ord, and so induces a monoidal structure on $\mathsf{Ord}_{\mathbb{N}}$, we will abuse notation by also denoting the induced monoidal structure on $\mathsf{Ord}_{\mathbb{N}}$ by $\oplus$, though in the later context we have $\underline{n} \oplus \underline{m} = \underline{n+m}$, rather than $\underline{n} \oplus \underline{m} \cong \underline{n+m}$.

Definition 9. A *2-fold port graph* consists of a pair of port graphs (Γ_v, Γ_h) with the same vertex set X. We will call a 2-fold port graph (Γ_v, Γ_h) an $\left(\begin{smallmatrix} & k & \\ m & & n \\ & \ell & \end{smallmatrix}\right)$ 2-fold port graph if Γ_h is an (m, n)-port graph and Γ_v is a (k, ℓ)-port graph.

Given an $\left(\begin{smallmatrix} & k & \\ m & & n \\ & \ell & \end{smallmatrix}\right)$ 2-fold port graph (Γ_v, Γ_h) with vertex set X, we can define an $(m + k, n + \ell)$-pre-port graph $\mathrm{Tot}(\Gamma_v, \Gamma_h)$ by setting

$$\mathrm{in}(x) = \mathrm{in}_v(x) \oplus \mathrm{in}_h(x)$$

and

$$\mathrm{out}(x) = \mathrm{out}_v(x) \oplus \mathrm{out}_h(x)$$

where $\oplus$ denotes the ordinal sum. We furthermore set

$$\iota(e) = \begin{cases} \iota_v(e) & e \in \underline{k} \amalg O_v \\ \iota_h(e) & e \in \underline{m} \amalg O_h, \end{cases}$$

where we are implicitly identifying, e.g., $\underline{m+k} \cong \underline{m} \oplus \underline{k}$.

Definition 10. A *double port graph* is a 2-fold port graph (Γ_v, Γ_h) such that $\mathrm{Tot}(\Gamma_v, \Gamma_h)$ is a port graph.

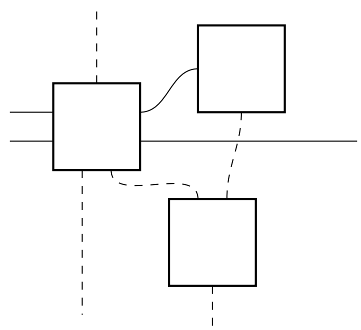
We can concatenate double port graphs (Γ_v, Γ_h) and (Ξ_v, Ξ_h)

- horizontally by setting
$$(\Gamma_v, \Gamma_h) \circ (\Xi_v, \Xi_h) = (\Gamma_v \amalg \Xi_v, \Gamma_h \circ \Xi_h),$$
- vertically by setting
$$(\Gamma_v, \Gamma_h) \bullet (\Xi_v, \Xi_h) = (\Gamma_v \circ \Xi_v, \Gamma_h \amalg \Xi_h)$$

whenever the right-hand side is well defined.

Definition 11. The *double category* DPG *of port graphs* has one object, horizontal and vertical morphism sets $\mathbb{N}$, and squares consisting of isomorphism classes of double port graphs.

Example 12. A typical double port graph looks as follows



We denote horizontal port graphs using solid wires and vertical ones using dashed wires. Vertical concatenation is performed with respect to the dashed wires, whereas horizontal concatenation is performed with respect to the solid wires. As in Example 5, we first horizontally concatenate the top two boxes and then vertically concatenate the resulting diagram with the bottom box.

2.3 Port labels

We now want to generalize our double category DPG to labeled double port graphs.

Definition 13. A *set of port labels* for a double port graph consists of a set $\mathcal{L}$ together with four functions
$$s^v, t^v, s^h, t^h : \mathcal{L} \longrightarrow \mathbb{N}.$$

An *$\mathcal{L}$-labeled double port graph* consists of a double port graph $(X, \mathrm{in}_v, \mathrm{out}_v, \mathrm{in}_h, \mathrm{out}_h, \iota_v, \iota_h)$ together with a function
$$\mathrm{lab} : X \longrightarrow \mathcal{L}$$

such that the diagram

$$\begin{array}{ccccc} \mathsf{Ord} & \xleftarrow{\mathrm{in}_a} & X & \xrightarrow{\mathrm{out}_a} & \mathsf{Ord} \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle \mathrm{lab}} & & \downarrow{\scriptstyle |-|} \\ \mathbb{N} & \xleftarrow[s^a]{} & \mathcal{L} & \xrightarrow[t^a]{} & \mathbb{N} \end{array}$$

commutes for each $a \in \{h, v\}$, where $| - |$ denotes the cardinality.

Example 14. Let D be a 1-object (strict) double category whose horizontal and vertical morphisms are given by $\mathbb{N}$. Denote by $\mathrm{Sq}(\mathsf{D})$ the set of squares of D. The horizontal and vertical source and target maps of D endow $\mathrm{Sq}(\mathsf{D})$ with the structure of a set of port labels. In particular, $\mathrm{Sq}(\mathsf{DPG})$ can itself be used to label double port graphs. We will employ such labeling in Section 2.4 when discussing the pasting operation.

Definition 15. Two labeled port graphs are said to be *isomorphic* if there exists an isomorphism of double port graphs between them that commutes with the labeling functions.

Notice that, if (Γ_h, Γ_v) and (Ξ_h, Ξ_v) are double port graphs with vertex sets X and Y respectively, and labeling functions lab^X and lab^Y valued in a set of port labels $\mathcal{L}$, the universal property of the coproduct determines a unique labeling

$$\mathrm{lab} : X \amalg Y \longrightarrow \mathcal{L}$$

on the disjoint union of Γ and Ξ compatible with the inclusions. If Γ and Ξ are composable (either horizontally or vertically), this yields a canonical $\mathcal{L}$-labeling of the composite.

Definition 16. The *double category* $\mathsf{DPG}_{\mathcal{L}}$ *of* $\mathcal{L}$*-labeled port graphs* has one object, horizontal and vertical morphism sets $\mathbb{N}$, and squares consisting of isomorphism classes of $\mathcal{L}$-labeled double port graphs.

2.4 Pasting double port graphs

We now define a pasting rule that allows us to glue a double port graph with the correct numbers of horizontal and vertical inputs and outputs into the place of a vertex of another double port graph.

On a formal level, let $\Gamma = (\Gamma_h, \Gamma_v)$ be an $\left(\begin{smallmatrix} & k & \\ m & & n \\ & \ell & \end{smallmatrix}\right)$-double port graph with vertex set X (and structure morphisms written with a superscript X), and let

$$\mathrm{lab} : X \longrightarrow \mathrm{Sq}(\mathsf{DPG})$$

be a labeling of Γ by squares of DPG.

We will denote a chosen representative of $\mathrm{lab}(x)$ by $\Lambda^x = (\Lambda^x_h, \Lambda^x_v)$ with set of vertices Y^x. The structure maps of Λ^x will be decorated with superscripts x, e.g., ι^x_h. Similarly, Λ^x will be a $\left(m^x \begin{smallmatrix} k^x \\ \\ \ell^x \end{smallmatrix} n^x \right)$-double port graph.

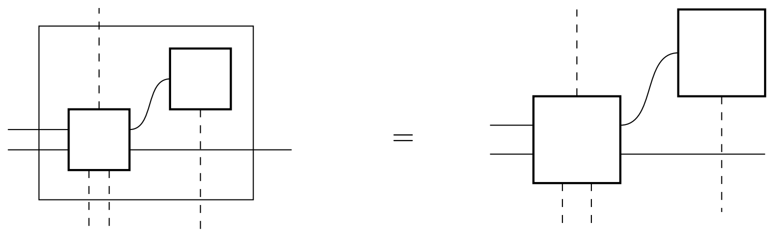
Given these data, we define a new $\left(m \begin{smallmatrix} k \\ \\ \ell \end{smallmatrix} n \right)$-double port graph $\Gamma \bullet \{\Lambda^x\}_{x \in X}$ as follows:

- The set of vertices of $\Gamma \bullet \{\Lambda^x\}_{x \in X}$ is
$$\coprod_{x \in X} Y^x.$$
- For every x, and every $y \in Y^x$ we define
$$\begin{aligned} \mathrm{in}_v(y) &= \mathrm{in}^x_v(y) \\ \mathrm{in}_h(y) &= \mathrm{in}^x_h(y) \\ \mathrm{out}_v(y) &= \mathrm{out}^x_v(y) \\ \mathrm{out}_h(y) &= \mathrm{out}^x_h(y) \end{aligned}$$
- The isomorphism ι_h is defined as follows. Note first that, for every $x \in X$, there is a unique order-preserving bijection $\alpha_x : \mathrm{in}_h(x) \cong \underline{m}^x$, and similarly $\beta_x : \mathrm{out}_h(x) \cong \underline{n}^x$. Define a relation on the set
$$\coprod_{x \in X} (\underline{m}^x \amalg O^x_h)$$
as follows. We say that $j \in \underline{m}^x$ is equivalent to $k \in O^z_h$ if $(\iota^X_h)^{-1} \alpha_x^{-1}(j) = \beta_z^{-1}(k)$. Note that, for $j \underline{m}^x$, there is either a unique $z \in X$ and a unique $k \in inO^z_h$ such that $j \sim k$, or $(\iota^X_h)^{-1}(\alpha(j)) = \ell \in \underline{m}$. By Lemma 18 below, the set $\left(\underline{m} \setminus (\iota^X_h)^{-1}(\underline{n})\right) \amalg \left(\coprod_{x \in X} O^x_h\right)$ is the quotient.

  An identical construction displays $\left(\underline{n} \setminus \iota^X_h(\underline{m})\right) \amalg \left(\coprod_{x \in X} I^x_h\right)$ as a quotient of
$$\coprod_{x \in X} (I^x_h \amalg \underline{n}^x).$$
  We then define ι_h to be the unique map such that the following diagram commutes.
$$\begin{array}{ccccccc} & & \coprod_{x \in X} (\underline{m}^x \amalg O^x_h) & \xrightarrow{\amalg_X \iota^x_h} & \coprod_{x \in X} (I^x_h \amalg \underline{n}^x) & & \\ & & \downarrow & & \downarrow & & \\ \underline{m} \amalg O_h & \cong & \underline{m} \amalg \left(\coprod_{x \in X} O^x_h\right) & \xrightarrow{\iota_h} & \underline{n} \amalg \left(\coprod_{x \in X} I^x_h\right) & \cong & \underline{n} \amalg I_h \\ & & \uparrow & & \uparrow & & \\ & & \underline{m} \cap (\iota^X_h)^{-1}(\underline{n}) & \xrightarrow[\iota^X_h]{} & \underline{n} \cap \iota^X_h(\underline{m}) & & \end{array} \tag{3}$$

The vertical structure is defined identically to the horizontal structure.

Example 17. The pasting operation is very natural when represented as boxes:



Here, Γ is the largest box, which is labeled by Λ^{x_1} and Λ^{x_2}, the medium sized box and the smallest boxes, respectively. The pasting $\Gamma \bullet \{\Lambda^{x_1}, \Lambda^{x_2}\}$ is the diagram on the right.

Lemma 18. *The quotient of*

$$\coprod_{x\in X} (\underline{m}^x \amalg O_h^x)$$

by the relation defined above is canonically isomorphic to $\big(\underline{m} \setminus (\iota_h^X)^{-1}(\underline{n})\big) \amalg \big(\coprod_{x\in X} O_h^x\big)$.

Proof. We define a map

$$\pi : \coprod_{x\in X} (\underline{m}^x \amalg O_h^x) \longrightarrow \big(\underline{m} \setminus (\iota_h^X)^{-1}(\underline{n})\big) \amalg \big(\textstyle\coprod_{x\in X} O_h^x\big)$$

as follows. On each copy of O_h^x, π acts as the identity. For $j \in \underline{m}^x$

$$\pi(j) := \begin{cases} k \in O_h^z & k \sim j \\ (\iota_h^X)^{-1}(\alpha_x^{-1}(j)) & \text{else} \end{cases}$$

This map is a quotient onto its image by construction. Moreover, for every $\ell \in \underline{m} \setminus (\iota_h^X)^{-1}(\underline{n})$, there is some $x \in X$ such that $\iota_h^X(\ell) \in \text{in}_h^X(x)$, and so the map is surjective as well, completing the proof. □

Proposition 19. *The construction above defines a double port graph* $\Gamma \bullet \{\Lambda^x\}_{x\in X}$.

Proof. We will show that the horizontal structure forms a port graph, and leave the remaining checks to the reader. Only two facts need to be checked: (1) ι_h is a bijection, and (2) the internal flow graph is acyclic.

As to the former, the top horizontal map in diagram (3) is an isomorphism, and both it and its inverse respect the equivalence relations by construction. Thus, it descends to an isomorphism

$$\big(\underline{m} \setminus (\iota_h^X)^{-1}(\underline{n})\big) \amalg \left(\coprod_{x\in X} O_h^x\right) \cong \big(\underline{n} \setminus \iota_h^X(\underline{m})\big) \amalg \left(\coprod_{x\in X} I_h^x\right).$$

Since the bottom map is simply an isomorphism on the complements of these subsets, it follows that ι_h is an isomorphism.

To see (2), suppose, to the contrary, there is an oriented cycle in the flow graph G of $\Gamma \bullet \{\Lambda^x\}_{x\in X}$. We will write this cycle as a chain of vertices and edges

$$y_0 \xrightarrow{e_1} y_1 \xrightarrow{e_2} \cdots \xrightarrow{e_k} y_k \xrightarrow{e_{k+1}} y_0.$$

Identifying the flow graphs H^x of the Λ^x with subgraphs of G, we can note that, were this cycle contained in any one of them, Λ^x would not be acyclic, a contradiction. We can thus choose a sequence $x_0, \ldots x_n \in X$ of vertices of Γ which partition the sequence $\{y_i\}$ by the subgraphs H^x. But then there is an oriented cycle

$$x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_n \longrightarrow x_0$$

in the flow graph of Γ, a contradiction. □

Proposition 20. *The operation $\bullet$ is associative up to isomorphism. That is*

$$\Gamma \bullet \left\{\Lambda^x \bullet \{\Xi^x_y\}_{y\in Y^x}\right\}_{x\in X} \cong \left(\Gamma \bullet \{\Lambda^x\}_{x\in X}\right) \bullet \{\Xi^x_y\}_{x\in X, y\in Y^x}.$$

Moreover, $\bullet$ is right-unital up to isomorphism, with unit given by the unique (up to isomorphism) collection of one-vertex port graphs forming a labeling of Γ.

2.5 Functoriality in port labels

We now turn to the various ways in which transformations of port labels can provide functors of double categories.

Definition 21. Let $\mathcal{L}, \mathcal{M}$ be sets of port labels. A *morphism of port labels* $\mathcal{L} \to \mathcal{M}$ is a map of sets

$$f : \mathcal{L} \longrightarrow \mathcal{M}$$

which commutes with the maps s^v, t^v, s^h, and t^h. We denote the category of sets of port labels with morphisms of port labels by PL.

Given a morphism $f : \mathcal{L} \to \mathcal{M}$ of port labels, we construct a double functor

$$\overline{f} : \mathsf{DPG}_{\mathcal{L}} \longrightarrow \mathsf{DPG}_{\mathcal{M}}$$

as follows. On objects, horizontal morphisms, and vertical morphisms, $\overline{f}$ is the identity. On squares, given a double port graph $(X, \mathrm{in}_v, \mathrm{out}_v, \mathrm{in}_h, \mathrm{out}_h, \iota_v, \iota_h)$ with $\mathcal{L}$-labels $\mathrm{lab} : X \to \mathcal{L}$, we send this to the same double port graph, but now with labels $f \circ \mathrm{lab} : X \to \mathcal{M}$. It is immediate that this is compatible with identities and composition, and thus defines a double functor. Similarly immediate is the following proposition.

Proposition 22. *The assignments* $\mathcal{L} \mapsto \mathsf{DPG}_{\mathcal{L}}$ *and* $(f : \mathcal{L} \to \mathcal{M}) \mapsto \overline{f}$ *define a functor*

$$\mathsf{DPG} : \mathsf{PL} \longrightarrow \mathsf{DCat}$$

to the category of double categories.

For any set $\mathcal{L}$ of port labels, the set $\mathrm{Sq}(\mathsf{DPG}_{\mathcal{L}})$ of squares of $\mathsf{DPG}_{\mathcal{L}}$ is itself a set of port labels, equipped with the horizontal and vertical source and target maps of $\mathsf{DPG}_{\mathcal{L}}$. Then, the canonical map

$$\iota : \mathcal{L} \longrightarrow \mathrm{Sq}(\mathsf{DPG}_{\mathcal{L}})$$

sending a label L to the double port graph Γ_L with a single L-labeled vertex defines a morphism of port labels. Consequently, there is a canonical double functor

$$\bar{\iota} : \mathsf{DPG}_{\mathcal{L}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{DPG}_{\mathcal{L}})}.$$

Next, we describe a construction that yields a double functor in the opposite direction.

Let us begin with a general observation. A morphism of labels $f : \mathcal{L} \to \mathrm{Sq}(\mathsf{D})$ induces a double functor

$$\overline{f} : \mathsf{DPG}_{\mathcal{L}} \longrightarrow \mathsf{D}. \tag{4}$$

In particular, for a double category D, we can consider $\mathrm{Sq}(\mathsf{D})$-labeled double port graphs. The resulting labeled port-graph category is equipped with a canonical double functor

$$\mathrm{can} : \mathsf{DPG}_{\mathrm{Sq}(\mathsf{D})} \longrightarrow \mathsf{D}.$$

That is, $\mathrm{can} = \overline{\mathrm{id}}$, corresponding to the identity morphism on $\mathrm{Sq}(\mathsf{D})$.

The pasting construction of the previous section provides an explicit description of this canonical double functor when $\mathsf{D} = \mathsf{DPG}_{\mathcal{L}}$. Given a square in $\mathsf{DPG}_{\mathrm{Sq}(\mathsf{DPG}_{\mathcal{L}})}$, that is, a double port graph Γ with labels represented by the $\mathcal{L}$-labeled double port graphs $\{\Lambda^x\}_{x\in X}$, we send this square to the double port graph

$$\Gamma \bullet \{\Lambda^x\}_{x\in X},$$

whose $\mathcal{L}$-labels are given by the labels of the Λ^x. This construction defines a functor

$$\mathrm{paste}_{\mathcal{L}} : \mathsf{DPG}_{\mathrm{Sq}(\mathsf{DPG}_{\mathcal{L}})} \longrightarrow \mathsf{DPG}_{\mathcal{L}}.$$

As a consequence, given Let a set $\mathcal{M}$ of port labels and a morphism of port labels

$$f : \mathcal{M} \longrightarrow \mathrm{Sq}(\mathsf{DPG}_{\mathcal{L}}),$$

we obtain a double composite functor

$$f_* : \mathsf{DPG}_{\mathcal{M}} \xrightarrow{\ \overline{f}\ } \mathsf{DPG}_{\mathrm{Sq}(\mathsf{DPG}_{\mathcal{L}})} \xrightarrow{\mathrm{paste}_{\mathcal{L}}} \mathsf{DPG}_{\mathcal{L}}. \tag{5}$$

3 Adaptive instruments

In this section, we introduce the double category of instruments. Throughout, we work with a weaker variant of double categories, namely *pseudo-double categories*, in which the horizontal composition is weakened; see Section A.

3.1 Instruments

For background on quantum theory, we refer to [7, 3]. A quantum system is described by a Hilbert space, which is typically taken to be finite-dimensional in quantum computing. We will have this restriction throughout the paper, so for us a Hilbert space will always be finite-dimensional. Hilbert spaces and linear maps between them can be assembled into a category, denoted by Hilb [3]. The tensor product $\otimes$ operation of Hilbert spaces endows this category with a symmetric monoidal structure.

For a Hilbert space V, we will write $L(V)$ to denote the space of linear maps (operators) on V. For a linear operator A, we will write $A^\dagger$ for its adjoint. The following operators are important in quantum theory: A is called

- *Hermitian* if $A = A^\dagger$,
- *positive semi-definite* if it can be written as $A = B^\dagger B$ for some operator B,
- *projection* if $A^2 = A$,
- *unitary* if its inverse is $A^\dagger$.

Positive-semidefinite operators are important in both representing quantum states and quantum measurements. A positive-semidefinite operators whose trace is 1 is called a *density operator*, also called *a quantum state*. A *quantum measurement* with outcome (output) set Y is given by a function (with finite support)

$$\Pi : Y \longrightarrow \mathrm{Pos}(V)$$

such that $\sum_{a \in Y} \Pi(a) = \mathbb{1}_V$. The quantum measurement is said to be *projective* if the function lands in the set of projectors. A quantum state ρ and a quantum measurement Π gives a probability distribution p defined by the *Born rule*:

$$p(a) = \mathrm{Tr}(\rho \Pi(a)).$$

A linear map $\phi : L(V) \to L(W)$ is called *positive* if it maps positive semi-definite operators to positive semi-definite operators. The map ϕ is called *completely positive* if $\phi \otimes \mathbb{1}_{L(W)}$ is positive for any Hilbert space W. A completely positive linear map is called a *channel* if it is trace-preserving. We can assemble channels into a category. We will write $\mathrm{CP}(V, W)$ and $\mathrm{C}(V, W)$ for completely positive linear maps and channels, respectively.

Definition 23. The *category of channels*, denoted by Chan, is defined as follows: its objects are Hilbert spaces, and its morphisms are channels between them.

Quantum measurements and more general quantum operations can be formulated using instruments. An *instrument* with outcome set Y is a function (finite support)

$$\Phi : Y \longrightarrow \mathrm{CP}(V, W)$$

such that $\sum_{a \in Y} \Phi^a$ is a channel. Here we adopt the notation that $\Phi^a = \Phi(a)$. Two important examples of instruments are given by (1) unitary transformations, and (2) quantum measurements. In the former case, the outcome set is a singleton and the channel is given by conjugating with the unitary. In the latter, the outcome set of the instrument is the same as the outcome set of the quantum measurement and the instrument is given by $\Phi^a(-) = \Pi(a)(-)\Pi(a)$. The Born rule generalizes to this case to give a probability distribution associated to a quantum state ρ and an instrument Φ:

$$p(a) = \mathrm{Tr}(\Phi^a(\rho)).$$

3.2 The double category

We now introduce the main objects of this section, namely the *adaptive instruments*, and construct the associated double category. More precisely, we work within the framework of pseudo-double categories (see Section A). Adaptive instruments serve as models for quantum computation [12, 22].

Definition 24. The *double category of instruments*, denoted by Inst, is a 1-object double category whose *vertical morphisms* are Hilbert spaces $\mathbb{C}[\underline{n}]$, where $n \geq 0$, and whose *horizontal morphisms* are sets. A *square* in Inst, bounded by the sets X, Y and the Hilbert spaces V, W, is of the form

$$\begin{array}{ccc} * & \xrightarrow{X} & * \\ {\scriptstyle V}\downarrow & \Phi & \downarrow{\scriptstyle W} \\ * & \xrightarrow[Y]{} & * \end{array}$$

and consists of a (finite-support) map

$$\Phi : X \times Y \longrightarrow \mathrm{CP}(V, W),$$

where $\mathrm{CP}(V, W)$ denotes the set of completely positive maps from $L(V)$ to $L(W)$. For each pair $(a, b) \in X \times Y$, we write Φ_a^b for the corresponding element of $\mathrm{CP}(V, W)$. Here X and Y are interpreted as the input and output (or outcome) sets, respectively. We additionally require that, for every $a \in X$, the sum

$$\sum_b \Phi_a^b : L(V) \longrightarrow L(W) \tag{6}$$

is a *channel*. The objects of this double category, that is, the functions Φ as defined above, are called *adaptive instruments*.

Remark 25. In this definition, we restrict to the skeletal subcategory of Hilb, namely the full subcategory $\mathsf{Hilb}_{\mathbb{N}}$ whose objects are $\mathbb{C}[\underline{n}]$ for $n \geq 0$. This forms a strict monoidal category with tensor product $\mathbb{C}[\underline{n}] \otimes \mathbb{C}[\underline{m}] = \mathbb{C}[\underline{n+m}]$. Accordingly, we obtain a pseudo-double category with strict vertical composition—given by this tensor product—and weak horizontal composition given by the Cartesian product of sets. We choose to keep the set direction weak, since the natural generalization to simplicial sets, which will appear in a later section, does not admit a canonical skeletal subcategory. In the orthogonal direction, one could instead work with the entire category Hilb to obtain a doubly weak double category in the sense of [23]. For simplicity, we prefer to strictify one direction. Of course, it would also be possible to strictify the horizontal direction by working with the skeletal subcategory $\mathsf{Fin}_{\mathbb{N}}$.

Each $\Phi_a := \Phi(a, -)$ above is an instrument in the usual sense. However, we have an additional input parameter. For the rest of this section will show this definition actually yields a well-defined double category.

Let us begin with defining the compositions. The horizontal composition of the diagrams

$$\begin{array}{ccc} * & \xrightarrow{X_1} & * \\ {\scriptstyle V}\big\downarrow & \Phi & \big\downarrow{\scriptstyle W} \\ * & \xrightarrow[Y_1]{} & * \end{array} \quad \text{and} \quad \begin{array}{ccc} * & \xrightarrow{X_2} & * \\ {\scriptstyle W}\big\downarrow & \Psi & \big\downarrow{\scriptstyle U} \\ * & \xrightarrow[Y_2]{} & * \end{array}$$

is the square $(\Psi \circ \Phi)$ defined by

$$(\Psi \circ \Phi)^{(b_1,b_2)}_{(a_1,a_2)} = \Psi^{b_2}_{a_2} \circ \Phi^{b_1}_{a_1}$$

Since the composition of completely positive operators is completely positive, this results in a completely positive operator. Moreover,

$$\begin{aligned} \sum_{(b_1,b_2)} (\Psi \circ \Phi)^{(b_1,b_2)}_{(a_1,a_2)} &= \sum_{(b_1,b_2)} \Psi^{b_2}_{a_2} \circ \Phi^{b_1}_{a_1} \\ &= \sum_{b_1} \sum_{b_2} \Psi^{b_2}_{a_2} \circ \Phi^{b_1}_{a_1} \\ &= \left(\sum_{b_2} \Psi^{b_2}_{a_2} \right) \circ \left(\sum_{b_1} \Phi^{b_1}_{a_1} \right) \end{aligned}$$

is a composite of channels, and is thus itself a channel. The identity for this horizontal composition of squares is the map

$$\mathrm{Id}^h_V : * \times * \longrightarrow \mathrm{CP}(V, V)$$

which picks out the identity map.

The vertical composition of

$$\begin{array}{ccc} * & \xrightarrow{X} & * \\ {\scriptstyle V_1}\big\downarrow & \Phi & \big\downarrow{\scriptstyle W_1} \\ * & \xrightarrow[Y]{} & * \end{array}$$

and

$$\begin{array}{ccc} * & \xrightarrow{Y} & * \\ {\scriptstyle V_2}\big\downarrow & \Psi & \big\downarrow{\scriptstyle W_2} \\ * & \xrightarrow[Z]{} & * \end{array}$$

is given by the map

$$\begin{aligned} (\Psi \bullet \Phi) : X \times Y &\longrightarrow \mathrm{CP}(V_1 \otimes V_2, W_1 \otimes W_2) \\ (a, c) &\longmapsto \textstyle\sum_b \Phi_a^b \otimes \Psi_b^c. \end{aligned}$$

Here, we use the identification $L(V_1) \otimes L(V_2) \cong L(V_1 \otimes V_2)$ obtained via the Kroenecker product. As above, this is clearly associative, since the Kroenecker product is, thus, we need to show that it is unital and yields completely positive operators which satisfy the desired channel condition. Complete positivity follows from the facts that the tensor product of two completely positive maps is still completely positive, and sum of completely positive maps remains to be so. Trace preservation of $\sum_c (\Psi \bullet \Phi)_a^c$ follows from the properties of trace.

As to unitality: The vertical identity morphism is defined by the square

$$\begin{aligned} \mathrm{Id}_X^v : X \times X &\longrightarrow \mathrm{CP}(\mathbb{C}, \mathbb{C}) \\ (a, b) &\longrightarrow \delta_{a,b} \mathbb{1}_{\mathbb{C}}. \end{aligned}$$

We then note that

$$(\Phi \bullet \mathrm{Id}_X^v)_a^c = \sum_{b \in X} \delta_{a,b} \mathbb{1}_{\mathbb{C}} \otimes \Phi_b^c = \Phi_a^c$$

and

$$(\mathrm{Id}_Y^v \bullet \Phi)_a^c = \sum_{b \in Y} \Phi_a^b \otimes (\delta_{b,c} \mathbb{1}_{\mathbb{C}}) = \Phi_a^c$$

showing that this is unital.

The horizontal identity morphism is defined by the square

$$\begin{aligned} \mathrm{Id}_V^h : * \times * &\longrightarrow \mathrm{CP}(V, V) \\ (*, *) &\longrightarrow \mathbb{1}_V. \end{aligned}$$

We similarly have $(\Phi \circ \mathrm{Id}^h_V) = (\mathrm{Id}^h_V \circ \Phi) = \Phi$. Furthermore, it is immediate that

$$\mathrm{Id}^v_* = \mathrm{Id}^h_{\mathbb{C}}\,.$$

The final check we need to perform is the interchange law.

Lemma 26. *Given the following diagram*

$$\begin{array}{ccccc}
* & \xrightarrow{X_1} & * & \xrightarrow{X_2} & * \\
{\scriptstyle V_1}\downarrow & & \downarrow{\scriptstyle W_1} & & \downarrow{\scriptstyle U_1} \\
* & \xrightarrow{Y_1} & * & \xrightarrow{Y_2} & * \\
{\scriptstyle V_2}\downarrow & & \downarrow{\scriptstyle W_2} & & \downarrow{\scriptstyle U_2} \\
* & \xrightarrow{Z_1} & * & \xrightarrow{Z_2} & *
\end{array}$$

with squares labeled by $\Phi_{i,j}$, where i is the row and j is the column, the interchange law holds:

$$((\Phi_{2,2} \circ \Phi_{2,1}) \bullet (\Phi_{1,2} \circ \Phi_{1,1})) = ((\Phi_{2,2} \bullet \Phi_{1,2}) \circ (\Phi_{2,1} \bullet \Phi_{1,1}))\,.$$

Proof. We compute

$$\begin{aligned}
((\Phi_{2,2} \circ \Phi_{2,1}) \bullet (\Phi_{1,2} \circ \Phi_{1,1}))^{(c_1,c_2)}_{(a_1,a_2)} &= \sum_{(b_1,b_2)} (\Phi_{1,2} \circ \Phi_{1,1})^{(b_1,b_2)}_{(a_1,a_2)} \otimes (\Phi_{2,2} \circ \Phi_{2,1})^{(c_1,c_2)}_{(b_1,b_2)} \\
&= \sum_{(b_1,b_2)} \left((\Phi_{1,2})^{b_2}_{a_2} \circ (\Phi_{1,1})^{b_1}_{a_1}\right) \otimes \left((\Phi_{2,2})^{c_2}_{b_2} \circ (\Phi_{2,1})^{c_1}_{b_1}\right) \\
&= \sum_{(b_1,b_2)} \left((\Phi_{1,2})^{b_2}_{a_2} \otimes (\Phi_{2,2})^{c_2}_{b_2}\right) \circ \left((\Phi_{1,1})^{b_1}_{a_1} \otimes (\Phi_{2,1})^{c_1}_{b_1}\right) \\
&= \sum_{b_1} \sum_{b_2} \left((\Phi_{1,2})^{b_2}_{a_2} \otimes (\Phi_{2,2})^{c_2}_{b_2}\right) \circ \left((\Phi_{1,1})^{b_1}_{a_1} \otimes (\Phi_{2,1})^{c_1}_{b_1}\right) \\
&= \left(\sum_{b_2} (\Phi_{1,2})^{b_2}_{a_2} \otimes (\Phi_{2,2})^{c_2}_{b_2}\right) \circ \left(\sum_{b_1} (\Phi_{1,1})^{b_1}_{a_1} \otimes (\Phi_{2,1})^{c_1}_{b_1}\right) \\
&= (\Phi_{2,2} \bullet \Phi_{1,2})^{c_2}_{a_2} \circ (\Phi_{2,1} \bullet \Phi_{1,1})^{c_1}_{a_1} \\
&= ((\Phi_{2,2} \bullet \Phi_{1,2}) \circ (\Phi_{2,1} \bullet \Phi_{1,1}))^{(c_1,c_2)}_{(a_1,a_2)}
\end{aligned}$$

Here, the only non-trivial manipulations we have used are (1) the interchange law for tensor products and composition, and (2), the bilinearity of composition with respect to sums. Thus, the interchange law holds. □

So we see that Definition 24 gives a well-defined 1-object double category.

3.3 The horizontal and vertical categories

Now we consider the horizontal and vertical monoidal categories associated with the 1-object double category of instruments (Definition 83).

The horizontal monoidal category $H(\mathsf{Inst})$ consists of objects given by Hilbert spaces of the form $\mathbb{C}[\underline{n}]$, $n \geq 0$. Its morphisms are obtained from squares by setting both the input X and output Y to be singletons. A square of the form

$$\begin{array}{ccc} * & \xrightarrow{*} & * \\ {\scriptstyle V}\big\downarrow & \Phi & \big\downarrow{\scriptstyle W} \\ * & \xrightarrow[*]{} & * \end{array}$$

corresponds to an instrument $* \times * \to \mathrm{CP}(V, W)$, equivalently to a quantum channel from V to W. The horizontal composition in $H(\mathsf{Inst})$ coincides with the composition of channels.

Proposition 27. *The horizontal monoidal category $H(\mathsf{Inst})$ can be identified with $(\mathsf{Chan}_{\mathbb{N}}, \otimes, \mathbb{C})$, the full subcategory of Chan on objects $\mathbb{C}[\underline{n}]$, $n \geq 0$.*

Definition 28. For a set Y, the set of probability distributions on Y is defined as

$$D(Y) := \left\{ p : Y \to \mathbb{R}_{\geq 0} : p \text{ finitely supported and } \sum_{a \in Y} p(a) = 1 \right\}.$$

This defines a functor $D : \mathsf{Set} \to \mathsf{Set}$, called the *distribution monad* [8].

The unit map $\delta : Y \to D(Y)$ is given by the Dirac (delta) distribution at each element, while the multiplication map $\mu : D^2(Y) \to D(Y)$ is induced by the multiplication in the semiring $\mathbb{R}_{\geq 0}$. The associated Kleisli category Set_D has sets as objects and morphisms of the form $X \to D(Y)$. It is symmetric monoidal: letting $m : D(Y) \times D(Y') \to D(Y \times Y')$ denote the map sending a pair of distributions (p, q) to their product,

$$p \cdot q(a, b) = p(a)\, q(b),$$

the tensor product $\odot$ in Set_D is given on objects by $Y \odot Y' = Y \times Y'$, and on morphisms by

$$p \odot q = m \circ (p \times q)$$

for $p : X \to D(Y)$ and $q : X' \to D(Y')$.

Now, turning to the vertical monoidal category $V(\mathsf{Inst})$, we observe that its objects are given by sets. A morphism in this category is represented by a square of the form

$$\begin{array}{ccc} * & \xrightarrow{X} & * \\ {\scriptstyle \mathbb{C}}\big\downarrow & \Phi & \big\downarrow{\scriptstyle \mathbb{C}} \\ * & \xrightarrow[Y]{} & * \end{array}$$

that is, by an instrument of the form $X \times Y \to \mathrm{CP}(\mathbb{C}, \mathbb{C})$. Since $\mathrm{CP}(\mathbb{C}, \mathbb{C}) \cong \mathbb{R}_{\geq 0}$, such instruments correspond precisely to Kleisli morphisms of the form $X \to D(Y)$.

Proposition 29. *The vertical monoidal category* $V(\mathsf{Inst})$ *can be identified with* $(\mathsf{Set}_D, \odot, *)$.

4 Quantum computation with qubits

Quantum computation with qubits is explained and recalled in this section. We introduce the double port graphs for Boolean circuits and quantum circuits, and construct the corresponding double functors into the double category of instruments, which endow these abstract port graphs with their operational meaning. In effect these constructions amount to describing universal gate sets for Boolean and quantum circuits, which is well-known [24]. We also prepare for later sections where *adaptivity* becomes central. Adaptivity plays a crucial role in quantum computation and has been extensively studied in the literature, with early examples including [25, 10, 26, 11]. Our formulation of adaptive quantum computation will be illustrated in the next section through prominent models.

4.1 Qubit instruments

A *qubit system* is represented by the Hilbert space $\mathbb{C}^2$. Taking an n-fold tensor product yields the Hilbert space $(\mathbb{C}^2)^{\otimes n}$ of an n-*qubit system*. We may identify $(\mathbb{C}^2)^{\otimes n}$ with the group algebra $\mathbb{C}[\mathbb{Z}_2^n]$, which in turn can be identified with $\mathbb{C}[\underline{2^n}]$ via the bijection $\mathbb{Z}_2^n \to \underline{2^n}$ sending each bit string $(x_1, \ldots, x_n)$ to the natural number $\sum_{i=1}^n 2^{n-i} x_i$.

Definition 30. The *double category of qubit instruments*, denoted by QBit, is the full double subcategory of the double category Inst of instruments whose squares are given by

$$\left((\mathbb{C}^2)^{\otimes m} \;\; \begin{matrix} \mathbb{Z}_2^k \\ \\ \mathbb{Z}_2^\ell \end{matrix} \;\; (\mathbb{C}^2)^{\otimes n} \right)$$

instruments. For convenience, we will simply write $\left(m \begin{matrix} k \\ \ell \end{matrix} n \right)$ for such a square, hence think of the vertical and horizontal morphisms as natural numbers.

Definition 31. We define the following categories:

- Bool denotes the full subcategory of Set whose objects are $\mathbb{Z}_2^n$, where $n \geq 0$.
- QChan denotes the full subcategory of Chan whose objects are qubit Hilbert spaces $(\mathbb{C}^2)^{\otimes n}$, where $n \geq 0$.

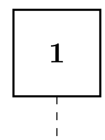
As a consequence of Propositions 27 and 29, the horizontal and vertical monoidal categories of QBit can be identified with QChan and Bool_D, respectively.

4.2 Boolean circuits

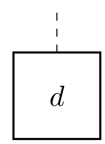
A function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$ is called a *Boolean map (or function)*. Such maps form the fundamental building blocks of computation over classical bits. In what follows, we construct a category of port graphs labeled by the primitive operations used to realize these maps.

Definition 32. The set $\mathcal{B}$ of *Boolean labels* consists of:

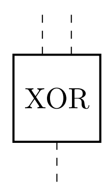
- The *1-state label*



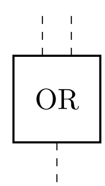
- The *delete label*



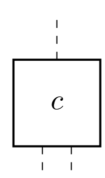
- The *XOR label*



- The *OR label*



- The *copy label*



The category of *Boolean-labeled port graphs* is $\mathsf{PG}_{\mathcal{B}}$.

Composition in this category is in the vertical direction. Treating this as a double category with no horizontal composition we can realize a double functor into instruments.

Definition 33. The double functor

$$\phi_{\mathcal{B}} : \mathsf{PG}_{\mathcal{B}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{QBit})} \xrightarrow{\text{can}} \mathsf{QBit}$$

is determined by the label morphism $\mathcal{B} \to \mathrm{Sq}(\mathsf{QBit})$ that maps the $\mathbf{1}$, d, XOR, OR, and c labels to the corresponding instruments, respectively, as follows:

- The $\mathbf{1}$-*state instrument*, for $s \in \mathbb{Z}_2$ and $\chi \in \mathbb{C}$,

$$\Phi_{\mathbf{1}}^{s}(\chi) := \delta_{s,1}\, \chi.$$

- The *delete instrument*, for $s \in \mathbb{Z}_2$ and $\chi \in \mathbb{C}$,

$$(\Phi_D)_s(\chi) := \chi.$$

- The *XOR instrument*, for $((s,r),t) \in \mathbb{Z}_2^2 \times \mathbb{Z}_2$ and $\chi \in \mathbb{C}$,

$$(\Phi_{\mathrm{XOR}})_{s,r}^{t}(\chi) := \delta_{t,s\oplus r}\, \chi$$

  where $\oplus$ denotes the mod-2 sum.

- The *OR instrument*, for $((s,r),t) \in \mathbb{Z}_2^2 \times \mathbb{Z}_2$ and $\chi \in \mathbb{C}$,

$$(\Phi_{\mathrm{OR}})_{s,r}^{t}(\chi) := \delta_{t,s\vee r}\, \chi$$

  where $\vee$ denotes the logical OR operation.

- The *copy instrument*, for $(r,(s,t)) \in \mathbb{Z}_2 \times \mathbb{Z}_2^2$ and $\chi \in \mathbb{C}$,

$$(\Phi_C)_r^{s,t}(\chi) := \delta_{r,s}\delta_{r,t}\, \chi.$$

Proposition 34. *The image of $\phi_{\mathcal{B}}$ can be identified by the category* Bool *of Boolean functions.*

We also consider the subset $\mathcal{B}_{\oplus} = \mathcal{B} \setminus \{\mathrm{OR}\}$ of *affine Boolean labels*, and the corresponding double functor from the category of *affine Boolean-labeled double port graphs*, obtained by restricting the label morphism above to this label set:

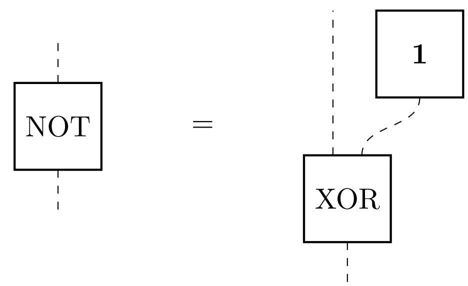
$$\phi_{\mathcal{B}_{\oplus}} : \mathsf{PG}_{\mathcal{B}_{\oplus}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{QBit})} \xrightarrow{\text{can}} \mathsf{QBit}.$$

Definition 35. The *category of affine Boolean maps*, denoted by Aff, is the subcategory of Bool with the same objects and morphisms given by affine Boolean maps, i.e., maps of the form $f(x) = a + \sum_{i=1}^{n} b_i x_i$ for some $a, b_i \in \mathbb{Z}_2$.

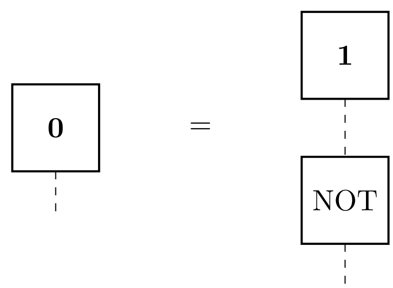
Proposition 36. *The image of $\phi_{\mathcal{B}_\oplus}$ can be identified by the category Aff of affine Boolean functions.*

In later sections, we will need the following labeled port graphs:

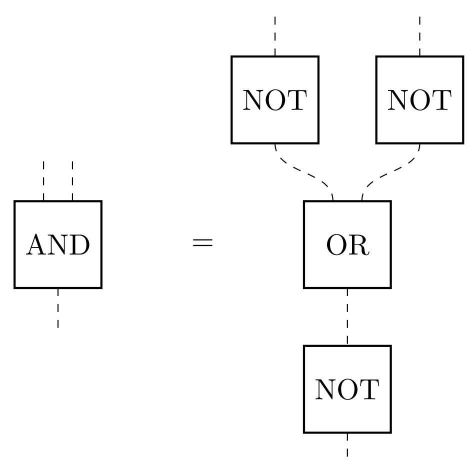
- *NOT label*:



- **0**-*state label*:



- *AND label*:



4.3 Quantum circuits

In the circuit model [9] a quantum computation is represented by a unitary operator

$$U : (\mathbb{C}^2)^{\otimes n} \longrightarrow (\mathbb{C}^2)^{\otimes n} \tag{7}$$

followed by a quantum measurement. Typically, the measurement is projective and the projection operators are usually taken to be those that project onto the canonical basis vectors. That is, writing

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and writing $|x_1 x_2 \dots x_n\rangle := |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$, where $x_i \in \{0, 1\}$, for the tensor product of these canonical basis vectors, the projectors of the measurement are given by the outer products

$$\Pi^x = |x_1 x_2 \dots x_n\rangle\langle x_1 x_2 \dots x_n|.$$

The outer product is understood as the usual matrix multiplication of a column vector with a row vector. The readout of a quantum circuit is described by a map

$$p : \mathbb{Z}_2^n \longrightarrow D(\mathbb{Z}_2^n).$$

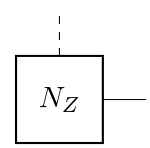
Denoting by p_r the probability distribution associated with an input $r \in \mathbb{Z}_2^n$ and by p_r^s the probability of observing the outcome s, we have

$$p_r^s = \mathrm{Tr}\big(\Pi^s \, U \, \Pi^r \, U^\dagger\big).$$

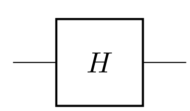
Typically, the default choice of input is the all-zero string $r = (0, 0, \dots, 0)$.

Definition 37. The set $\mathcal{C}$ of *QC labels* consists of:

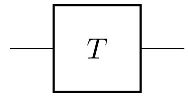
- The *Z-preparation label*



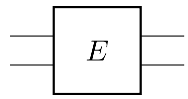
- The *Hadamard label*



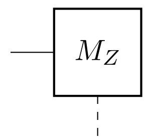
- The *T-gate label*



- The *entangling label*



- The *(destructive) Z-measurement label*



The double category of *QC-labeled double port graphs* is $\mathsf{DPG}_{\mathcal{C}}$.

To define a double functor into the double category of instruments, we recall some basic quantum operations. The Pauli matrices are given by

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

It is straightforward to verify that these matrices are both Hermitian and unitary. The standard, or computational, vectors $|0\rangle, |1\rangle \in \mathbb{C}^2$ are eigenvectors of the Pauli Z operator, satisfying $Z|x\rangle = (-1)^x|x\rangle$ for $x \in \mathbb{Z}_2$. A measurement in the Z-basis is described by the projectors

$$\Pi_Z^r := |r\rangle\langle r| = \tfrac{1}{2}(\mathbb{1} + (-1)^r Z). \tag{9}$$

The Hadamard and T gates are given by

$$H = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix},$$

and the controlled-Z gate by

$$E = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z.$$

Definition 38. The double functor

$$\phi_{\mathcal{C}} : \mathsf{DPG}_{\mathcal{C}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{QBit})} \xrightarrow{\text{can}} \mathsf{QBit}$$

is determined by the label morphism $\mathcal{C} \to \mathrm{Sq}(\mathsf{QBit})$ that maps the N_Z, H, T, E, and M_Z labels to the corresponding instruments:

- The *Z-state preparation instrument*, for $r \in \mathbb{Z}_2$ and $\chi \in \mathbb{C}$,

$$(\Phi_{N_Z})_r(\chi) = \chi |r\rangle\langle r|.$$

- The *U-gate instruments*,

$$\Phi_U(-) = U(-)U^\dagger, \tag{10}$$

  where $U \in \{H, T, E\}$.

- The *(destructive) Z-measurement instrument*, for $s \in \mathbb{Z}_2$,

$$\Phi^s_{M_Z}(-) = \mathrm{Tr}(\Pi^s_Z(-)\Pi^s_Z).$$

The double category determined by the image of $\phi_{\mathcal{C}}$ will be denoted by QC.

A fundamental result in quantum computing, the Solovay–Kitaev theorem [9] establishing *quantum universality*, can be formulated as a structural property of this category. To this end, we introduce a subcategory of the category of quantum channels that captures the unitaries realizable by quantum circuits, thereby allowing us to express quantum universality within the framework of port graphs.

Definition 39. Let QUni denote the full subcategory of Uni whose objects are qubit Hilbert spaces.

Hence, a morphism in this category is precisely a map of the form described in Eq. (7).

Theorem 40 (Universality)**.** *Let $H(\mathsf{QC})$ denote the horizontal monoidal category of QC. Then, for each $n \geq 0$, the group*

$$G_n := H(\mathsf{QC})(n, n)$$

is a dense subgroup of the unitary group $U\big((\mathbb{C}^2)^{\otimes n}\big)$.

The vertical direction corresponds to the notion of *reversible computation*, which can be formalized in categorical terms as follows.

Definition 41. Let Rev denote the subcategory of Bool having the same objects, but whose morphisms are precisely the bijective Boolean maps.

Proposition 42. *The vertical monoidal category can be identified as*

$$V(\mathsf{QC}) \;=\; \mathsf{Rev}_D.$$

4.4 Adaptive local instruments

As preparation for the adaptive computational models and the local operations, we introduce the notion of *adaptive local instruments*. This concept will also be useful in formulating the computational power of instruments in Section 6.

An instrument is said to be *local* if it is constructed from single-qubit instruments together with Boolean operations. These Boolean operations implement *adaptivity*, that is, the dependence of an instrument's inputs on the outputs of other instruments. The Boolean component may include all Boolean operations or a restricted subclass; in our setting, we restrict to affine Boolean operations, as these constitute the natural choice for standard models of adaptive quantum computation. More precisely, a single-qubit instrument Φ can be regarded as an n-qubit instrument by applying Φ to a specific qubit while leaving the remaining qubits unaffected. For $1 \leq i \leq n$, the instrument acting on the ith qubit is defined by

$$\Phi^{(i)} = \mathbb{1}_{L(\mathbb{C}^2)} \otimes \cdots \otimes \underbrace{\Phi}_{i\text{th position}} \otimes \cdots \otimes \mathbb{1}_{L(\mathbb{C}^2)}.$$

Instruments of this form will be referred to as *single-qubit instruments.* Let $\mathcal{S}$ denote the subset of squares in $\mathrm{Sq}(\mathsf{QBit})$ consisting of single-qubit $\left(\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix}\right)$-instruments. The inclusion $\mathcal{S} \to \mathrm{Sq}(\mathsf{QBit})$ of labels induces a double functor

$$\phi_{\mathcal{S}} : \mathsf{DPG}_{\mathcal{S}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{QBit})} \xrightarrow{\text{can}} \mathsf{QBit}.$$

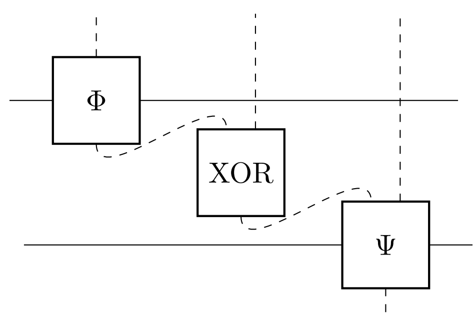
We will write QBit_S for the image of the double functor $\phi_{\mathcal{S}}$.

We combine the single-qubit instrument labels with affine Boolean labels to form the label set $\mathcal{A} = \mathcal{S} \amalg \mathcal{B}_{\oplus}$. This label set is equipped with a morphism $\mathcal{A} \to \mathrm{Sq}(\mathsf{QBit})$ induced by the inclusion of the single-qubit instruments and the affine Boolean instruments. This yields a double functor

$$\phi_{\mathcal{A}} : \mathsf{DPG}_{\mathcal{A}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{QBit})} \xrightarrow{\text{can}} \mathsf{QBit}.$$

The image of $\phi_{\mathcal{A}}$ will be denoted by QBit_A.

Example 43. The Boolean part allows us to implement adaptivity:



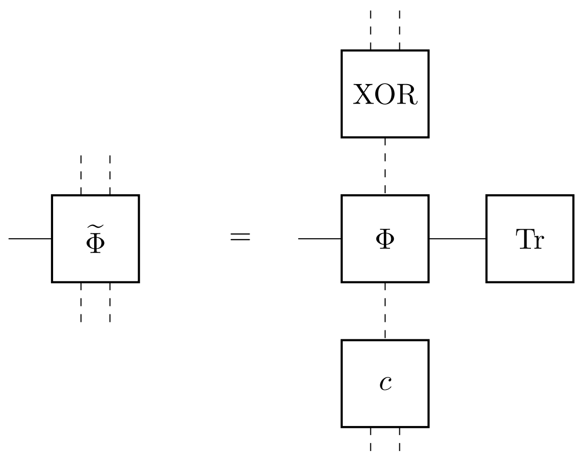
where the input to Φ is defined as the XOR of the output of Φ with an auxiliary input.

The various double subcategories of QBit assemble into a commutative diagram of double functors

$$\begin{array}{ccccc} & & \mathsf{Aff}_{\oplus} & \hookrightarrow & \mathsf{Bool} \\ & & \downarrow & & \downarrow \\ \mathsf{QBit}_S & \hookrightarrow & \mathsf{QBit}_A & \hookrightarrow & \mathsf{QBit}. \end{array}$$

We will encounter prominent models of quantum computation based on adaptive computation (Sections 5.1, 5.2, and 5.3). Later, in Section 6, we will analyze the computational power of these adaptive models when restricted to the vertical direction—that is, their classical computational power expressed in terms of Kleisli morphisms. For this purpose, we introduce a modification, referred to as *Bell instruments*, in which the input and output sets coincide.

For each single-qubit instrument $\Phi \in \mathcal{S}$ let us define the instrument $\widetilde{\Phi} : \mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \to \mathrm{CP}(\mathbb{C}^2, \mathbb{C})$ by



We define a new label set consisting of the single-qubit labels

$$\widetilde{\mathcal{S}} = \{\, \widetilde{\Phi} \mid \Phi \in \mathcal{S} \,\}.$$

Let $\widetilde{\mathcal{A}}$ denote the label set obtained as the union of $\mathcal{B}_{\oplus}$ and $\widetilde{\mathcal{S}}$. The natural inclusion $\widetilde{\mathcal{A}} = \widetilde{\mathcal{S}} \amalg \mathcal{B}_{\oplus} \to \mathrm{Sq}(\mathsf{QBit})$ then induces a double functor

$$\phi_{\widetilde{\mathcal{A}}} : \mathsf{DPG}_{\widetilde{\mathcal{A}}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{QBit})} \xrightarrow{\text{can}} \mathsf{QBit}.$$

Definition 44. An instrument in QC_A is called a *Bell instrument* if it lies in the image of a $\widetilde{\mathcal{A}}$-labeled double port graph X such that, for each instrument $\Phi \in \widetilde{\mathcal{S}}$ with vertical inputs (r_1, r_2) and outputs (s_1, s_2) appearing as a label of X, the following conditions hold:

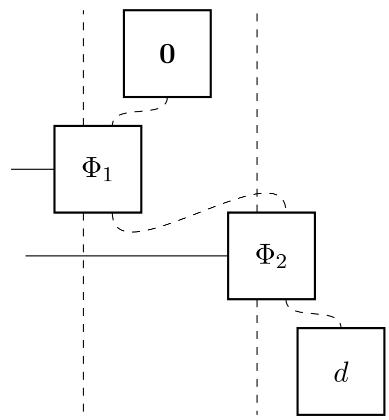
- The input r_1 connects only to the input wire of X and not to any other instrument labels, while r_2 either connects to other instrument labels or originates from a **0**-state label.
- The output s_1 connects only to the output wire of X and not to any other instrument labels, while s_2 either connects to other instrument labels or terminates at a delete label.

A Bell instrument is of type

$$\left(m \overset{k}{\underset{k}{\,}} m{-}k \right),$$

where the vertical outputs correspond to the left vertical outputs of the k single-qubit instruments used. When $k = m$, such an instrument is called an *m-qubit Bell instrument.*

Example 45. A typical $\left(2 \overset{2}{\underset{2}{\,}} 0 \right)$ Bell instrument has the form



5 Adaptive quantum computation

In this section, we present prominent adaptive models of quantum computation from a double categorical perspective, and describe the relationships among them in terms of double functors. The formulation of these models in terms of quantum instruments was first introduced in [12] and subsequently refined in [22].

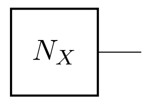
5.1 Measurement-based quantum computation

Measurement-based quantum computation (MBQC) is a computational framework introduced by Raussendorf–Briegel [10], in which computation is performed through a sequence of adaptive quantum measurements. In this section, we describe this model using our double categorical language.

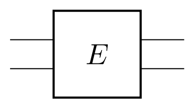
We follow the computational model known as the *measurement calculus*, introduced in [27]. This framework specifies the basic quantum operators of MBQC, which we will use as labels for the double port graphs.

Definition 46. The set $\mathcal{M}$ of *MBQC labels* consists of the affine Boolean labels $\mathcal{B}_{\oplus}$ together with:

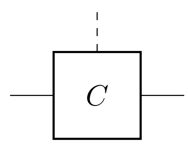
- The *X-preparation label*



- The *entangling label*

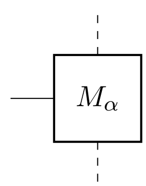


- The *C-correction labels*



  where $C = X$ or Z.

- The *(destructive) α-measurement labels*



  where $\alpha \in [0, 2\pi)$.

The double category of *MBQC-labelled double port graphs* is $\mathsf{DPG}_{\mathcal{M}}$.

For the definition of the double functor into QBit, we will need some preliminary definitions. Consider the vectors of the form

$$|+^r_\alpha\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^r e^{i\alpha}\,|1\rangle\right),$$

where $\alpha \in [0, 2\pi)$. We will use the corresponding projectors

$$\Pi_\alpha^r := |+_\alpha^r\rangle \langle +_\alpha^r| = \frac{1}{2}\left(I + (-1)^r(\cos\alpha\, X + \sin\alpha\, Y)\right). \tag{11}$$

In particular, the eigenvectors of the Pauli X operator satisfy $X|\pm\rangle = \pm|\pm\rangle$, where $|+\rangle := \left|+_0^0\right\rangle$ and $|-\rangle := \left|+_0^1\right\rangle$.

Definition 47. The double functor

$$\phi_{\mathcal{M}} : \mathsf{DPG}_{\mathcal{M}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{QBit})} \xrightarrow{\text{can}} \mathsf{QBit}$$

is determined by the label morphism $\mathcal{M} \to \mathrm{Sq}(\mathsf{QBit})$ that sends the labels N_X, E, C, and M_α to the corresponding instruments:

- The *X-preparation instrument*, for $\chi \in \mathbb{C}$,

$$\Phi_N(\chi) = \chi|+\rangle\langle+|.$$

- The *E-gate instrument* Φ_E, see Eq. (10).
- The *C-correction instruments*, for $r \in \mathbb{Z}_2$,

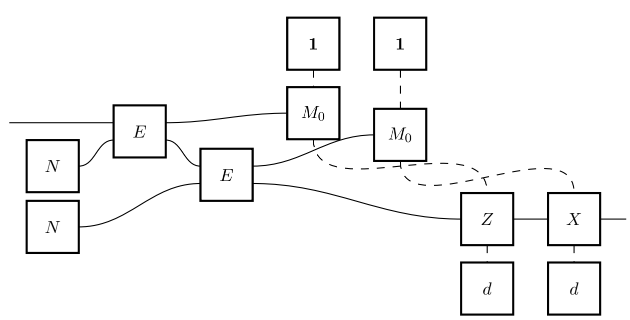
$$(\Phi_C)_r(-) = C^r(-)C^r, \tag{12}$$

  where $C \in \{X, Z\}$.
- The *(destructive) α-measurement instrument*, for $(r, s) \in \mathbb{Z}_2^2$,

$$(\Phi_{M_\alpha})_r^s(-) = \mathrm{Tr}\big(\Pi^s_{(-1)^r\alpha}(-)\Pi^s_{(-1)^r\alpha}\big).$$

Computing with measurements, as in MBQC, is inspired by an important quantum computation protocol called *teleportation*. Next, we describe this in our double categorical framework.

Example 48. The teleportation protocol [27] can be implemented by a $\mathcal{M}$-labeled $\left({}_1\,{}^0_0\,{}_1\right)$-double port graph of the form



Under the $\phi_{\mathcal{M}}$ double functor this labeled port graph is sent to an instrument which implements the identity channel on a single qubit.

As a consequence of the Solovay–Kitaev theorem (Theorem 40), quantum computational power can be achieved using a finite universal gate set. The counterpart of this result in MBQC is that one can likewise restrict to a finite set of measurement operations.

Definition 49. Let $\mathcal{M}[\pi/4]$ denote the subset of $\mathcal{M}$ where α is restricted to $0, \pm\pi/4$:

$$\mathcal{M}[\pi/4] = \{N_Z, E, C, M_\alpha \colon C = X, Z \text{ and } \alpha = 0, \pm\pi/4\}.$$

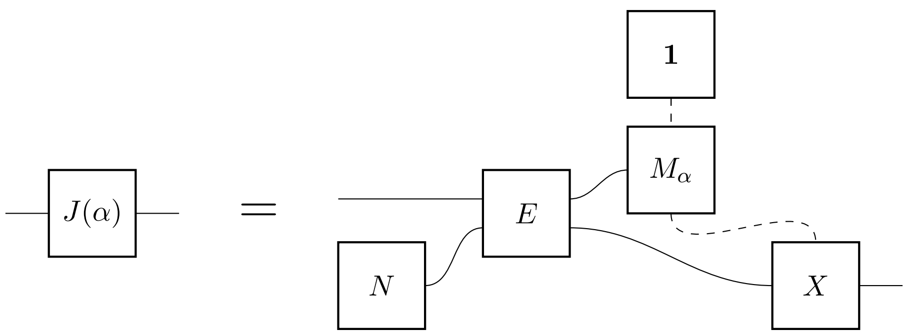
The double category determined by the image of the restriction of $\phi_{\mathcal{M}}$ to $\mathsf{DPG}_{\mathcal{M}[\pi/4]}$ will be denoted by MBQC.

The QC model can be compared to the MBQC model. We begin by constructing morphisms of port labels:

$$\varphi_{\mathcal{C},\mathcal{M}[\pi/4]} : \mathcal{C} \longrightarrow \mathrm{Sq}(\mathsf{DPG}_{\mathcal{M}[\pi/4]}).$$

For the definition of φ we need the following fundamental $\mathcal{M}[\pi/4]$-labeled double port graph; see [27]:

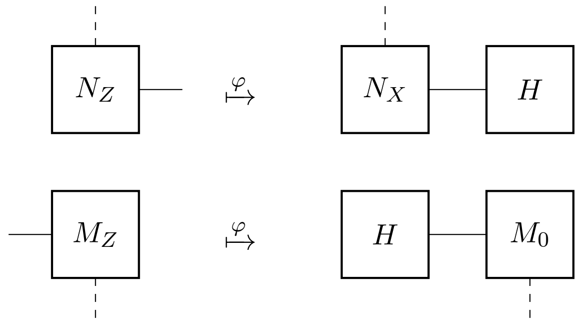
Definition 50. The $J(\alpha)$-*gadget* is defined to be the following $\mathcal{M}$-labeled double port graph:



Then, φ is defined on the gate labels as follows:

$$\varphi(U) = \begin{cases} J(0) & U = H \\ J(0) \circ J(\pi/4) & U = T \\ E & U = E. \end{cases}$$

Here, we identify each label in the target with the double port graph with a single vertex labeled by that label. The composition of labels $M \circ L$ stands for the double port graph $\Gamma_M \circ \Gamma_L$. We will write H and T for the $\mathcal{M}[\pi/4]$-labeled double port graphs $J(0)$ and $J(0) \circ J(\pi/4)$, respectively. The remaining labels can be defined as follows:



Now, the pasting construction of Section 2.4 and the functoriality of label morphisms as discussed in Section 2.5 gives a double functor

$$\kappa_{\mathcal{C},\mathcal{M}[\pi.4]} : \mathsf{DPGM}_{\mathcal{C}} \xrightarrow{\overline{\varphi}} \mathsf{DPG}_{\mathrm{Sq}(\mathsf{DPG}_{\mathcal{M}[\pi/4]})} \xrightarrow{\text{paste}} \mathsf{DPG}_{\mathcal{M}[\pi/4]}. \tag{13}$$

This formalizes what is meant by a quantum circuit implemented by an MBQC. Furthermore, the operational meaning is preserved; that is, we have a commutative diagram of double functors

$$\begin{array}{ccc} \mathsf{DPG}_{\mathcal{C}} & \xrightarrow{\kappa} & \mathsf{DPG}_{\mathcal{M}[\pi/4]} \\ & \underset{\phi}{\searrow} & \downarrow{\scriptstyle \phi} \\ & & \mathsf{QBit} \end{array} \tag{14}$$

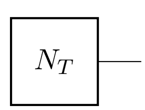
Typically, an MBQC consists of a state preparation followed by a sequence of adaptive measurements that together implement the computation. This structure is formalized in Proposition 84 via the notion of a *standard form*, which expresses the rewrite rules of [27] in terms of double categorical composition.

5.2 Quantum computation with magic states

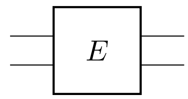
In this section, we introduce a double category for quantum computation with magic states (QCM), one of the well-known models of universal quantum computation introduced in [11].

Definition 51. The set $\mathcal{Q}$ of *QCM labels* consists of the affine Boolean labels together with:

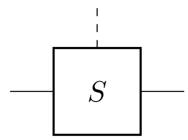
- The *T-preparation label*



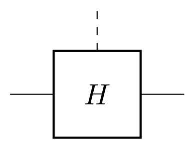
- The *entangling label*



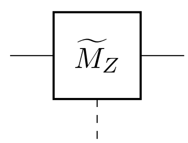
- The *S-correction label*



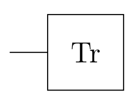
- The *H-correction label*



- The *(nondestructive) Z-measurement label*



- The *trace label*



The double category of *QCM-labeled double port graphs* is $\mathsf{DPG}_{\mathcal{Q}}$

Definition 52. We define a double functor

$$\phi_{\mathcal{Q}} : \mathsf{DPG}_{\mathcal{Q}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{Qbit})} \xrightarrow{\text{can}} \mathsf{QBit}$$

induced by the label morphism $\mathcal{Q} \to \mathrm{Sq}(\mathsf{Qbit})$ that sends the N_T, E, S/H, $\widetilde{M}_Z$, Tr labels to the respective instruments:

- The *T-preparation instrument*, for $\chi \in \mathbb{C}$,

$$\Phi_{N_T}(\chi) = \chi T|{+}\rangle\langle{+}|T^\dagger.$$

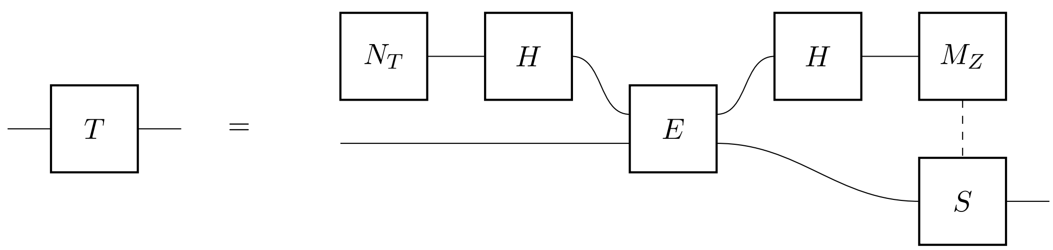
- The *E-gate instrument:* Φ_E, see Eq. (10).
- The *C-correction instruments* with $C = S$ or H, see Eq. (12).
- The *(non-destructive) measurement instrument*, for $r \in \mathbb{Z}_2$,

$$(\Phi_{\widetilde{M_Z}})^r(-) = \Pi_Z^r(-)\Pi_Z^r$$

  where Π_Z^r is as given in Eq. (9).
- The *trace instrument* given by the trace map $\mathrm{Tr} : L(\mathbb{C}^2) \to (\mathbb{C})$.

The affine Boolean labels are sent to their corresponding Boolean instruments.

Definition 53. The *T-gadget* (see [28]) is defined to be the following $\mathcal{Q}$-labeled double port graph:



The double category determined by the image of $\phi_{\mathcal{Q}}$ will be denoted by QCM.

We construct a morphism of labels

$$\varphi_{\mathcal{C},\mathcal{Q}} : \mathcal{C} \longrightarrow \mathrm{Sq}(\mathsf{DPG}_{\mathcal{Q}})$$

by sending H-label to the vertical composition $H \bullet \mathbf{1}$ and sending the T-label to the T-gadget. The remaining labels are mapped to those with the same names. Now, using the pasting operation (Section 2.4) we can construct the double functor

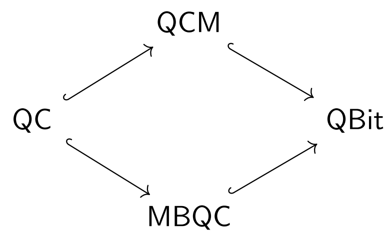
$$\kappa_{\mathcal{C},\mathcal{Q}} : \mathsf{DPG}_{\mathcal{C}} \xrightarrow{\overline{\varphi}} \mathsf{DPG}_{\mathrm{Sq}(\mathsf{DPG}_{\mathcal{Q}})} \xrightarrow{\text{paste}} \mathsf{DPG}_{\mathcal{Q}}. \tag{15}$$

This formalizes conversion of a quantum circuit (QC) into the QCM model.

Similar to the MBQC case, we have a commutative diagram of double functors

$$\begin{array}{ccc} \mathsf{DPG}_{\mathcal{C}} & \xrightarrow{\kappa} & \mathsf{DPG}_{\mathcal{Q}} \\ & \searrow^{\phi} & \downarrow{\scriptstyle \phi} \\ & & \mathsf{QBit} \end{array} \tag{16}$$

Diagrams (14) and (16) combine to yield (Diagram 1). The corresponding double subcategories assembles into



(17)

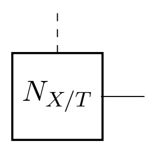
Further conversions are possible between the models QC, QCM, and MBQC; their explicit formulation is left to the reader.

5.3 Measurement-based Pauli computation

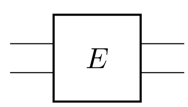
Now, we specialize to the measurement-based computational model introduced in [13] and further developed in [12]. We begin with the latter version, which involves fewer basic operations. We call this model measurement-based Pauli computation (MBPC).

Definition 54. The set $\mathcal{P}$ of *MBPC labels* consists of the affine Boolean labels together with::

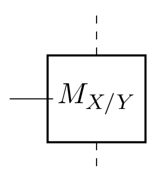
- The *X/T-preparation label*



- The *entangling label*

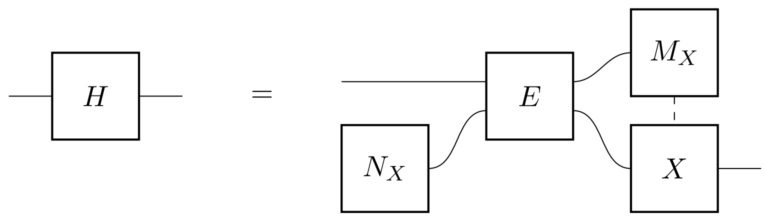


- The *(destructive) X/Y-measurement label*



The double category of *MBPC-labeled double port graphs* is $\mathsf{DPG}_{\mathcal{P}}$

We begin by describing the H-gate label, a special case of the $J(\alpha)$-gadget (Definition 50) with $\alpha = 0$:



Here, and henceforth we write M_X or M_Y for the vertical composition of $M_{X/Y}$ with the **0-** or **1**-state label. Similarly, we write N_X and N_T for the preparation labels obtained from $N_{X/T}$.

Definition 55. We define a double functor

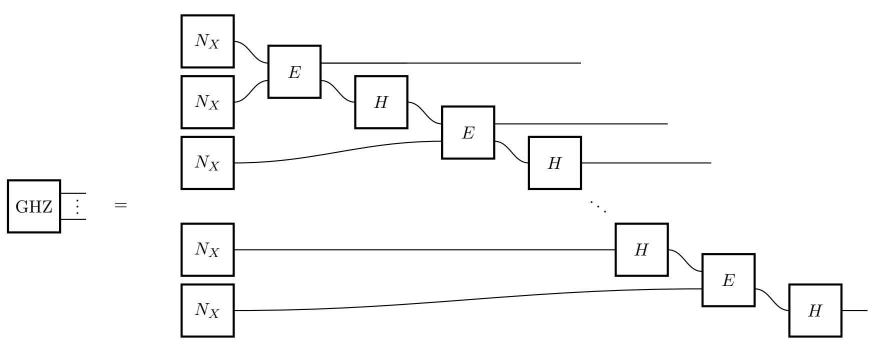
$$\phi_{\mathcal{P}} : \mathsf{DPG}_{\mathcal{P}} \longrightarrow \mathsf{DPG}_{\mathrm{Sq}(\mathsf{Qbit})} \xrightarrow{\text{can}} \mathsf{QBit}$$

induced by the label morphism $\mathcal{P} \to \mathrm{Sq}(\mathsf{Qbit})$ that sends the $N_{X/T}$, E, and $M_{X/Y}$ labels to the respective instruments:

- The *X/T-preparation instrument*, for $\chi \in \mathbb{C}$,
$$(\Phi_{N_{X/T}})_s(\chi) = \chi T^s |+\rangle\langle+| (T^s)^\dagger.$$
- The *E-gate instrument:* Φ_E, see Eq. (10).
- The *(destructive) X/Y-measurement instrument*, for $s, r \in \mathbb{Z}_2$,
$$(\Phi_{M_{X/Y}})_s^r(-) = \mathrm{Tr}(S^s \Pi_X^r (S^s)^\dagger (-) S^s \Pi_X^r (S^s)^\dagger)$$
where $\Pi_X^r = (\mathbb{1} + (-1)^r X)/2$, which also coincides with Π_α in Eq. (11) when $\alpha = 0$.

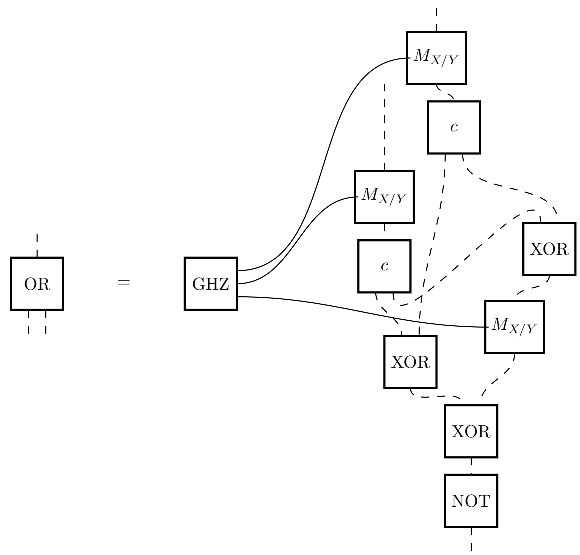
The affine Boolean labels are sent to their corresponding Boolean instruments. The double category determined by the image of $\phi_{\mathcal{P}}$ will be denoted by MBPC.

A key construction in this model is the realization of the OR-gate label by affine Boolean labels and quantum labels. This construction is crucial in understanding the vertical direction of the double category $\mathsf{DPG}_{\mathcal{P}}$. For this construction we need the Greenberger–Horne–Zeilinger (GHZ) state [27] for n-qubits, which in $\mathsf{DPG}_{\mathcal{P}}$ can be implemented as follows:



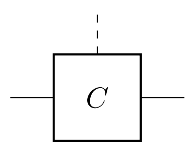
For the implementation of the OR-gate label we use the following gadget (see [15, 16]).

Definition 56. The *OR-gadget* is defined by



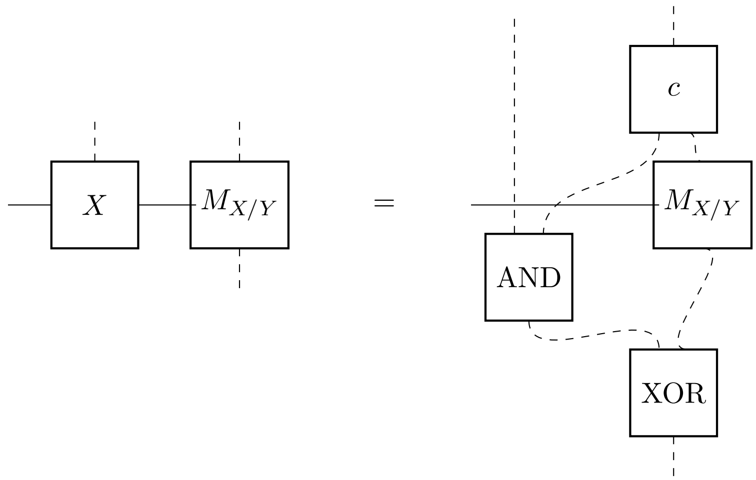
We also introduce an extended version of the label set $\mathcal{P}$ by including corrections. This model was introduced in [13].

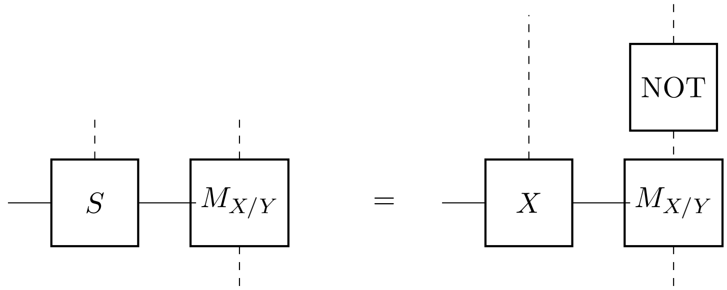
Definition 57. The set $\widetilde{\mathcal{P}}$ of *MBPC labels with corrections* consists of the labels in $\mathcal{P}$ together with the *C-correction labels*



where $C = X$ or S.

In the extended version we have the following *correction vs measurement rewrite rules*:





The second rewrite rule can be further expanded using the first one. Observe that the crucial point is that the AND-label appears, which is not an affine Boolean operation. However, as we have seen in Section 4.2 the AND-label can be implemented using the OR-label, which in turn can be implemented in $\mathsf{DPG}_{\mathcal{P}} \subset \mathsf{DPG}_{\widetilde{\mathcal{P}}}$ using the OR-gadget.

Definition 58. We extend the double functor $\phi_{\mathcal{P}}$ of Definition 55 to form a commutative diagram of double functors

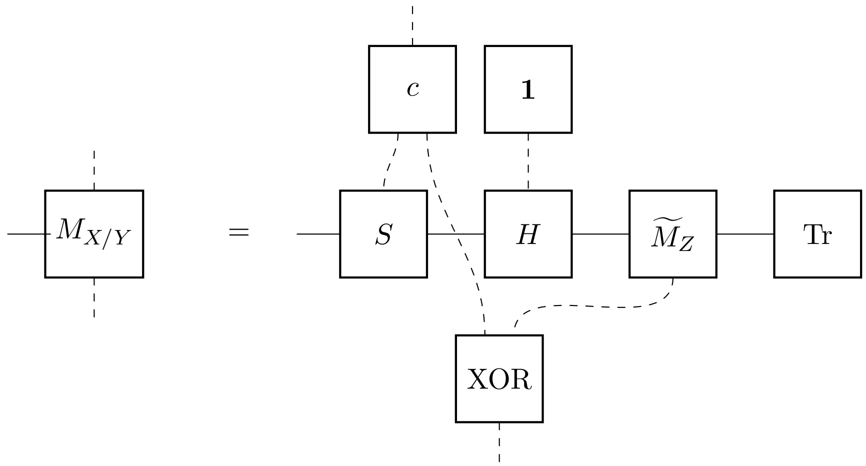
$$\begin{array}{ccc} \mathsf{DPG}_{\widetilde{\mathcal{P}}} & \xrightarrow{\ \phi_{\widetilde{\mathcal{P}}}\ } & \mathsf{QBit} \\ \big\uparrow & \nearrow_{\phi_{\mathcal{P}}} & \\ \mathsf{DPG}_{\mathcal{P}} & & \end{array}$$

by extending the corresponding label morphism to $\widetilde{\mathcal{P}} \to \mathrm{Sq}(\mathsf{Qbit})$ by sending the remaining C-correction labels to corresponding C-gate instrument 10. The double category determined by the image of $\phi_{\widetilde{\mathcal{P}}}$ will be denoted by MBPC_c.

To compare MBPC with corrections to QCM we define a morphism of port labels

$$\varphi_{\widetilde{\mathcal{P}},\mathcal{Q}} : \widetilde{\mathcal{P}} \longrightarrow \mathrm{Sq}(\mathsf{DPG}_{\mathcal{Q}})$$

by sending N_T, E, S labels to the labels with the same name, and by sending H-correction label to the H-correction described above and

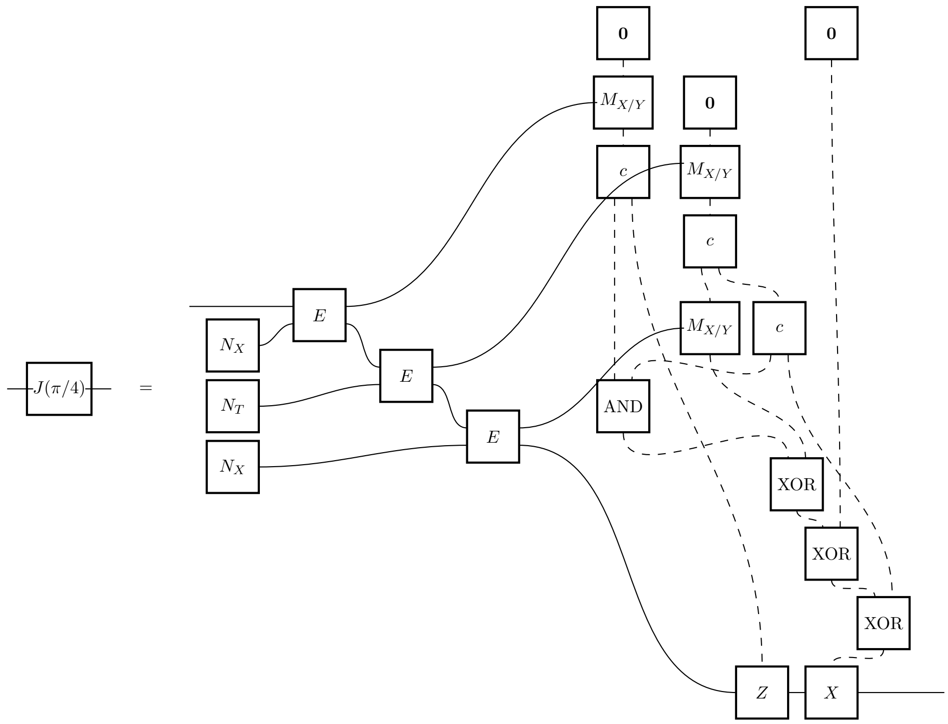


Composing with the pasting double functor we obtain

$$\kappa_{\tilde{\mathcal{P}},\mathcal{Q}} : \mathsf{DPG}_{\widetilde{\mathcal{P}}} \xrightarrow{\overline{\varphi}} \mathsf{DPG}_{\mathrm{Sq}(\mathsf{DPG}_{\mathcal{Q}})} \xrightarrow{\text{paste}} \mathsf{DPG}_{\mathcal{Q}}. \tag{18}$$

To compare the image of $\mathsf{DPG}_{\mathcal{C}}$ in $\mathsf{DPG}_{\mathcal{Q}}$ we compose the $J(\pi/4)$-gadget (Definition 50 with $\alpha = \pi/4$) with the teleportation diagram (Example 48) and apply the rewrite rules of Section B to obtain the *teleported* $J(\pi/4)$*-gadget*:

Definition 59. The *teleported* $J(\pi/4)$*-gadget* is defined by



Each occurrence of the AND operation can again be implemented using the OR-gadget. The Z-correction label can be implemented using the S-correction label. Then we can implement the T-gate label using the labels in $\mathcal{P}$ by the horizontal composition $H \circ J(\pi/4)$. Similarly the preparation and measurement labels in $\mathcal{C}$ can be implemented using the labels in $\widetilde{\mathcal{P}}$. This way we obtain a morphism of labels

$$\varphi_{\mathcal{C},\widetilde{\mathcal{P}}} : \mathcal{C} \longrightarrow \mathrm{Sq}(\mathsf{DPG}_{\widetilde{\mathcal{P}}})$$

and in turn we obtain a double functor

$$\kappa_{\mathcal{C},\widetilde{\mathcal{P}}} : \mathsf{DPG}_{\mathcal{C}} \xrightarrow{\overline{\varphi}} \mathsf{DPG}_{\mathrm{Sq}(\mathsf{DPG}_{\widetilde{\mathcal{P}}})} \xrightarrow{\text{paste}} \mathsf{DPG}_{\widetilde{\mathcal{P}}} \tag{19}$$

which allows us to implement a quantum circuit in the MBPC-with-corrections model.

Diagrams (18) and (19) assemble into Diagram (2) and the corresponding double subcategories are given by

$$\begin{array}{ccccc} & & \mathsf{QC} & \hookrightarrow & \mathsf{QBit} \\ & & \downarrow & & \uparrow \\ \mathsf{MBPC} & \hookrightarrow & \mathsf{MBPC}_c & \hookrightarrow & \mathsf{QCM} \end{array} \tag{20}$$

Proposition 60. *Let* D *denote one of the double categories* MBPC*,* MBPC_c*,* MBQC*, or* QCM*. Then,*

$$V(\mathsf{D}) = \mathsf{Bool}_D$$

and each $H(\mathsf{D})(n,n)$ *is a dense subgroup of* $U((\mathbb{C}^2)^{\otimes n})$ *for* $n \geq 0$*.*

Proof. Follows from Diagrams 17 and 20 together with Proposition 36, Theorem 40, and Definition 56. □

6 Power of adaptive quantum computation

In this section, we study the computation of Boolean functions using adaptive instruments. It is known that quantum contextuality plays a crucial role in determining the corresponding computational power [15, 16, 17, 29, 20, 19]. We extend these approaches to computations modeled by a simplicial variant of adaptive instruments, which we call *simplicial instruments*. These objects also form a double category, where the vertical input/output is given by simplicial sets. This construction connects naturally to the framework of *simplicial distributions* introduced in [14] for the study of contextuality. In this way, quantitative results on computational power can be directly related to corresponding degrees of contextuality.

6.1 Simplicial instruments

Our constructions in this section provide a natural extension of the notions of simplicial distributions and measurements introduced in [14] to the setting of instruments. This, in turn, allows us to define the notion of contextuality for instruments.

Definition 61. For Hilbert spaces V and W, the functor $\mathcal{CP}_{V,W} : \mathsf{Set} \to \mathsf{Set}$ is defined as follows: On a set X, it gives the set of instruments

$$\mathcal{CP}_{V,W}(X) = \left\{ \Phi : X \to \mathrm{CP}(X,Y) \colon \Phi \text{ finitely supported}, \sum_{x \in X} \Phi^x \in \mathrm{C}(V,W) \right\}.$$

For a set map $f : X \to Y$, the corresponding map $f_* : \mathcal{CP}_{V,W}(X) \to \mathcal{CP}_{V,W}(Y)$ is defined by

$$(f_*(\Phi))^y = \sum_{x \in X \colon f(x)=y} \Phi^x.$$

The key idea of simplicial instruments is to upgrade the input and output sets to simplicial sets (spaces). Let Δ denote the *simplex category* with objects $[n] = \{0, 1, \cdots, n\}$ where $n \geq 0$ and morphisms given by ordinal maps, i.e., order-preserving functions. Note that this category can be identified[1] with the skeleton of Ord.

[1] Note that $[n-1] \cong \underline{n}$.

Definition 62. A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$. It is convenient to write X_n for the set $X([n])$, which represents the set of n-simplices. The simplex category is generated by the coface and codegeneracy maps so using this fact we can think of a simplicial set as a collection of sets $\{X_n\}_{n\geq 0}$ together with the face $d_i : X_n \to X_{n-1}$ and degeneracy $s_j : X_n \to X_{n+1}$ maps. The *category* sSet *of simplicial sets* is the functor category consisting of functors $\Delta^{\mathrm{op}} \to \mathsf{Set}$.

An n-simplex $x \in X_n$ is called *non-degenerate* if it does not lie in the image of any degeneracy map. A simplicial set map $f : X \to Y$ is completely determined by its restriction to the non-degenerate simplices of X. When X is *finite-dimensional*, meaning that it has finitely many non-degenerate simplices, we may work with this finite set of simplices. A non-degenerate n-simplex is called *generating* if it does not lie in the image of any face map.

The functor $\mathcal{CP}_{V,W}$ lifts to the category of simplicial sets to yield a functor $\mathcal{CP}_{V,W} : \mathsf{sSet} \to \mathsf{sSet}$ by level-wise application.

Definition 63. A *simplicial instrument* with input simplicial set X, output simplicial set Y, and Hilbert spaces V and W is a simplicial set map

$$\Phi : X \longrightarrow \mathcal{CP}_{V,W}(Y).$$

The collection of such instruments forms the *double category of simplicial instruments*, denoted by sInst. It is the one-object double category whose vertical morphisms are Hilbert spaces, horizontal morphisms are simplicial sets, and whose *squares* are given by simplicial instruments:

$$\begin{array}{ccc} * & \xrightarrow{X} & * \\ {\scriptstyle V}\downarrow & \Phi & \downarrow{\scriptstyle W} \\ * & \xrightarrow[Y]{} & * \end{array}$$

The double subcategory of simplicial qubit instruments will be denoted by sQBit.

To verify that we have indeed a well-defined double category we verify that the constructions of Section 3.2, including horizontal/vertical compositions and units, respect the simplicial structure.

Lemma 64. *The horizontal and vertical composition of simplicial instruments, defined levelwise by the corresponding horizontal and vertical compositions of adaptive instruments, is simplicial.*

Proof. Let $\theta : [m] \to [n]$ be an ordinal map. We check that the horizontal and the vertical compositions are simplicial. First, consider $\Phi : X_1 \to \mathcal{CP}_{V,W}(Y_1)$ and $\Psi : X_2 \to \mathcal{CP}_{W,U}(Y_2)$.

The horizontal composition rule is simplicial:

$$\begin{aligned}
(\theta^*(\Psi \circ \Phi))^{(y_1,y_2)}_{(x_1,x_2)} &= \sum_{(y_1',y_2')\,:\; \theta^*(y_1',y_2')=(y_1,y_2)} (\Phi \circ \Psi)^{(y_1',y_2')}_{(x_1,x_2)} \\
&= \sum_{(y_1',y_2')\,:\; \theta^*(y_1',y_2')=(y_1,y_2)} \Psi^{y_2'}_{x_2} \circ \Phi^{y_1'}_{x_1} \\
&= \left(\sum_{y_2'\,:\; \theta^*(y_2')=y_2} \Psi^{y_2'}_{x_2} \right) \circ \left(\sum_{y_1'\,:\; \theta^*(y_1')=y_1} \Phi^{y_1'}_{x_1} \right) \\
&= \Psi^{y_2}_{\theta^*(x_2)} \circ \Phi^{y_1}_{\theta^*(x_1)} \\
&= (\Psi \circ \Phi)^{(y_1,y_2)}_{\theta^*(x_1,x_2)}.
\end{aligned}$$

Next, take $\Phi : X \to \mathcal{CP}_{V_1,W_1}(Y)$ and $\Psi : Y \to \mathcal{CP}_{V_2,W_2}(Z)$. The vertical composition rule is also simplicial:

$$\begin{aligned}
(\theta^*(\Psi \bullet \Phi))^z_x &= \sum_{z'\,:\; \theta^*(z')=z} (\Psi \bullet \Phi)^{z'}_x \\
&= \sum_{z'\,:\; \theta^*(z')=z} \sum_y \Phi^y_x \otimes \Psi^{z'}_y \\
&= \sum_y \Phi^y_x \otimes \left(\sum_{z'\,:\; \theta^*(z')=z} \Psi^{z'}_y \right) \\
&= \sum_y \Phi^y_x \otimes \Psi^z_{\theta^*(y)} \\
&= \sum_{y'} \sum_{y\,:\; \theta^*(y)=y'} \Phi^y_x \otimes \Psi^z_{y'} \\
&= \sum_{y'} \left(\sum_{y\,:\; \theta^*(y)=y'} \Phi^y_x \right) \otimes \Psi^z_{y'} \\
&= \sum_{y'} \Phi^{y'}_{\theta^*(x)} \otimes \Psi^z_{y'} \\
&= (\Phi \bullet \Psi)^z_{\theta^*(x)}
\end{aligned}$$

□

The identity morphisms are obtained directly from the corresponding formulas for instruments, since these are automatically simplicial. The vertical identity morphism Id^v_X :

$X \to \mathcal{CP}_{\mathbb{C},\mathbb{C}}(X)$ is given by $x \mapsto (x' \mapsto \delta_{x,x'}\,\mathbb{1}_{\mathbb{C}})$, while the horizontal identity morphism $\mathrm{Id}^h_V : \Delta^0 \to \mathcal{CP}_{V,V}(\Delta^0)$ is determined by the identity operator $\mathbb{1}_V$. As before, these satisfy $\mathrm{Id}^v_* = \mathrm{Id}^h_{\mathbb{C}}$.

Finally, the interchange law holds since it holds at each level by Lemma 26. Thus, sInt forms a well-defined double category.

6.1.1 Simplicial distributions

In the horizontal direction, a

$$\left(V \overset{\Delta^0}{\underset{\Delta^0}{}} W \right)\text{-square in } \mathsf{sInst}$$

is precisely a channel, since Δ^0 carries no simplicial information—it is the simplicial set consisting of a single element $*$ in each degree, with identity face and degeneracy maps as its simplicial structure. In contrast, in the vertical direction we no longer land in the Kleisli category of sets Set_D, but rather in its simplicial analogue sSet_D [14, 30], the Kleisli category of simplicial sets. The distribution monad D extends to a monad on sSet via levelwise application, thereby allowing the formation of this Kleisli category.

Proposition 65. *The horizontal and vertical monoidal categories of* sInst *can be identified with* Chan *and* sSet_D*, respectively.*

We write $\delta : X \to D(X)$ for the unit of this monad, which maps each simplex to the delta distribution concentrated at that simplex.

Definition 66. A *simplicial distribution* is a simplicial set map of the form

$$p : X \longrightarrow D(Y)$$

in the Kleisli category sSet_D. A simplicial distribution is said to be *deterministic* if it arises as the composite

$$\delta^f : X \xrightarrow{f} Y \xrightarrow{\delta} D(Y)$$

for some simplicial set map $f : X \to Y$. This construction extends to a convex map[2]

$$\Theta : D(\mathsf{sSet}(X, Y)) \longrightarrow \mathsf{sSet}(X, D(Y)),$$

which embeds convex combinations of deterministic simplicial maps into simplicial distributions. A simplicial distribution is called *contextual* if it does not lie in the image of Θ; otherwise, it is said to be *non-contextual*.

[2]That is, a map preserving convex combinations.

There are different ways to construct the map Θ. One approach is to define it explicitly by formula, as done in [14]; alternatively, it can be derived from the fact that $\mathsf{sSet}(X, D(Y))$ forms a convex set, that is, a D-algebra, as shown in [30].

As a notational convention, for a simplicial set map $f : X \to Y$, we write f_σ for the simplex in Y_n corresponding to an n-simplex $\sigma \in X_n$. For a simplicial distribution $p : X \to D(Y)$, the symbol p_σ denotes the distribution in $D(Y_n)$ associated with the n-simplex σ.

Definition 67. The *support* of a simplicial distribution $p : X \to D(Y)$, denoted by $\mathrm{supp}(p)$, consists of those simplicial set maps $f : X \to Y$ satisfying $p_\sigma(f_\sigma) > 0$ for all simplices $\sigma \in X_n$ and all $n \geq 0$. A simplicial distribution is said to be *strongly contextual* if its support is empty.

As expected, strong contextuality implies contextuality [14, Proposition 5.1]. This notion can be refined by introducing quantitative degrees of contextuality for simplicial distributions [30].

Definition 68. The *non-contextual fraction* of a simplicial distribution $p : X \to D(Y)$, denoted by $\mathrm{NCF}(p)$, is defined as the supremum of $\lambda \in [0, 1]$ such that

$$p = \lambda q + (1 - \lambda)q',$$

where q and q' range over simplicial distributions, with q non-contextual. The corresponding *contextual fraction* is defined by $\mathrm{CF}(p) = 1 - \mathrm{NCF}(p)$.

Unraveling the definition, we observe that p is strongly contextual if and only if $\mathrm{NCF}(p) = 0$.

6.2 Simplicial Bell scenarios

The double category of instruments Inst (QBit) can be viewed as a double subcategory of the double category sInst (sQBit) of simplicial instruments, when the input and output sets are regarded as discrete simplicial sets. In this section, we present a construction that embeds the double category of adaptive instruments QBit_A, introduced in Section 4.4, into the framework of simplicial instruments via *simplicial Bell scenarios*. In the context of quantum foundations, Bell scenarios constitute canonical examples of contextuality scenarios [31, 32, 33, 34]. The key insight underlying this embedding is the *locality* property of adaptive instruments.

For the description of simplicial Bell scenarios, we need a construction from simplicial sets.

Definition 69. Given simplicial sets X and Y, their join [35], denoted by $X \star Y$, is the simplicial set with n-simplices

$$(X \star Y)_n = X_n \amalg (\amalg_{p+1+q=n} X_p \times X_q) \amalg Y_n.$$

The simplicial structure on the first and the last factors are induced by those of X and Y, respectively. In the middle factor, for $(\sigma, \tau) \in X_p \times Y_q$,

$$d_i(\sigma, \tau) = \begin{cases} (d_i(\sigma), \tau) & i \leq p,\ p \neq 0 \\ (\sigma, d_{i-1-p}(\tau)) & i > p,\ q \neq 0 \end{cases} \tag{21}$$

and the degeneracy maps are defined similarly (obtained by replacing d with s). For the boundary cases, when $p = 0$ we set $d_0(\sigma, \tau) = \tau$, and when $q = 0$ we set $d_m(\sigma, \tau) = \sigma$.

Now, we introduce the following simplicial sets:

- Let S^0 denote the $\Delta^0 \amalg \Delta^0$, the coproduct of two copies of the 0-zero simplex. For $m \in \mathbb{N}$, we define the *m-sphere* as the join

$$S^m := \underbrace{S^0 \star S^0 \star \cdots \star S^0}_{m}.$$

- For $\ell \in \mathbb{N}$, we define the simplicial set Υ^ℓ with n-simplices given by

$$(\Upsilon^\ell)_n := \mathsf{FSet}([n], [\ell])$$

  and the simplicial structure is obtained by precomposing with the ordinal maps. We will specialize to $\ell = 1$.

We begin with an explicit description of S^m. Let us write σ_0 and σ_1 for the two 0-simplices of S^0. Then, by the definition of the join construction we can define m-simplices of S^m, denoted by $\sigma_{i_1 \dots i_m}$, for each $(i_1, \dots, i_m) \in \mathbb{Z}_2^m$, by

$$\sigma_{i_1 \dots i_m} = (\sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_m}).$$

These are precisely the generating simplices of S^m. Using the simplicial structure of the join we can assemble these simplices into a simplicial set map

$$\pi_m : \mathbb{Z}_2^m \times \Delta^m \longrightarrow S^m.$$

Given a simplicial distribution $p : S^m \to D(\Upsilon^1)$, we consider the diagram

$$\begin{array}{ccc} \mathbb{Z}_2^m \times \Delta^m & \xrightarrow{\pi_m} & S^m \\ \downarrow{\scriptstyle \bar{p} \times \mathrm{Id}_{\Delta^m}} & & \downarrow{\scriptstyle p} \\ D(\mathbb{Z}_2^m) \times \Delta^m & \xrightarrow{\cong} & D(\Upsilon^1) \end{array} \tag{22}$$

for some Kleisli map $\bar{p} : \mathbb{Z}_2^m \to D(\mathbb{Z}_2^m)$.

Definition 70. Given a simplicial distribution $p : S^m \to D(\Upsilon^1)$ and a function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2^m$, we say that p *computes* f if the associated Kleisli map $\bar{p}$ coincides with f, that is, $\bar{p} = f$.

Lemma 71. *Simplicial set maps $g : S^m \to \Upsilon^1$ are in bijective correspondence with functions*

$$g_0 : \amalg^m \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2.$$

Proof. We begin by observing that Υ^1 is isomorphic to the nerve of a category. Let C denote the category with object set $\mathbb{Z}_2$ and, for each pair of objects $a, b \in \mathbb{Z}_2$, a unique morphism $c : a \to b$ labeled by $c \in \mathbb{Z}_2$, satisfying $b = a + c \bmod 2$. Then the simplicial map $\Upsilon^1 \to N(\mathsf{C})$ defined in degree 1 by $(a, b) \mapsto c$ is an isomorphism of simplicial sets [36, Lemma 3.30].

It is a standard fact that a simplicial map with target the nerve of a category is completely determined by its action on 1-simplices and their face relations. For S^m, each non-degenerate 1-simplex is uniquely determined by its boundary vertices. Hence, specifying a simplicial map $g : S^m \to \Upsilon^1$ amounts to specifying a set map on vertices, that is, a function $g_0 : (S^m)_0 \to (\Upsilon^1)_0 \cong \mathbb{Z}_2$. Identifying $(S^m)_0$ with $\coprod^m \mathbb{Z}_2$ establishes the desired correspondence. □

Proposition 72. *Let $p : X \to D(\Upsilon^1)$ be a deterministic simplicial distribution. Then the associated Kleisli map $\bar{p} : \mathbb{Z}_2^m \to D(\mathbb{Z}_2^m)$ is of the form*

$$\bar{p} : \mathbb{Z}_2^m \xrightarrow{\ f\ } \mathbb{Z}_2^m \xrightarrow{\ \delta\ } D(\mathbb{Z}_2^m)$$

for some affine function f.

Proof. Since p is deterministic, it is of the form $\delta \circ s$ for some simplicial set map $s : S^m \to \Upsilon^1$. By Lemma 71, s is determined by its restriction to 0-simplices, namely by a function $s_0 : \coprod^m \mathbb{Z}_2 \to \mathbb{Z}_2$. Equivalently, we may view s_0 as a family of functions $(s_0)_i : \mathbb{Z}_2 \to \mathbb{Z}_2$ for $1 \le i \le m$. Then the diagram (22) factors as

$$\begin{array}{ccc}
\mathbb{Z}_2^m \times \Delta^m & \xrightarrow{\quad \pi_m \quad} & S^m \\
\Big\downarrow{\scriptstyle \bar{s}\times \mathrm{Id}} & & \Big\downarrow{\scriptstyle s} \\
\mathbb{Z}_2^m \times \Delta^m & \xrightarrow{\quad \cong \quad} & \Upsilon^1 \\
\Big\downarrow{\scriptstyle \delta\times \mathrm{Id}} & & \Big\downarrow{\scriptstyle \delta} \\
D(\mathbb{Z}_2^m) \times \Delta^m & \xrightarrow{\quad \cong \quad} & D(\Upsilon^1)
\end{array}$$

where the function $\bar{s} : \mathbb{Z}_2^m \to \mathbb{Z}_2^m$ is given by

$$\bar{s}(a_1, \ldots, a_m) = \big((s_0)_1(a_1), \ldots, (s_0)_m(a_m)\big).$$

Since every function $\mathbb{Z}_2 \to \mathbb{Z}_2$ is affine Boolean, it follows that $\bar{s}$, and hence f, is affine. □

Construction 73. Given an m-qubit Bell instrument Φ (Definition 44), we construct a simplicial instrument

$$\hat{\Phi} : S^m \longrightarrow \mathcal{CP}_{(\mathbb{C}^2)^{\otimes m}, \mathbb{C}}(\Upsilon^1)$$

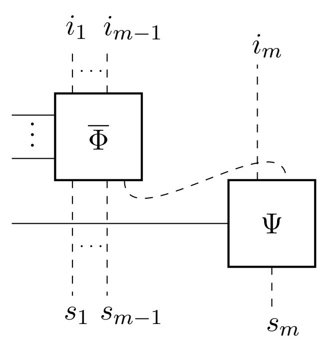
by assigning to each non-degenerate simplex $\sigma_{i_1 \ldots i_m} : \Delta^m \to S^m$, corresponding to $(i_1, \ldots, i_m) \in \mathbb{Z}_2^m$, the instrument $\Phi_{i_1 \ldots i_m}$.

This construction gives a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_2^m \times \Delta^m & \xrightarrow{\quad \pi_m \quad} & S^m \\ \Big\downarrow {\scriptstyle \Phi \times \mathrm{Id}_{\Delta^m}} & & \Big\downarrow {\scriptstyle \hat{\Phi}} \\ \mathcal{CP}_{(\mathbb{C}^2)^{\otimes m},\mathbb{C}}(\mathbb{Z}_2^m) \times \Delta^m & \longrightarrow & \mathcal{CP}_{(\mathbb{C}^2)^{\otimes m},\mathbb{C}}(\Upsilon^1). \end{array}$$

Lemma 74. *The assignment* $\sigma_{i_1\ldots i_m} \mapsto \Phi_{i_1\ldots i_m}$ *is simplicial.*

Proof. Let $\Phi_1, \ldots, \Phi_m$ denote the single-qubit instruments that appear in Φ. Since the internal flow graph of the underlying port graph is acyclic (see Definition 3), at least one of these instruments is such that its outcome is not used to control the remaining ones. Otherwise, we can create a loop in the port graph violating acyclicity. We can assume that Φ_m satisfies this property. Then, the instrument Φ has the form



where $\overline{\Phi}$ consists of $\Phi_1, \ldots, \Phi_{m-1}$ with the adaptive Boolean part controlling the input to Φ_m, and Ψ is the vertical composition of Φ_m with d-label terminating its right-hand output. We will do induction on m. For $m = 0$, the statement holds trivially for S^0 (no non-trivial simplicial identities). For $m \geq 1$, assume that the statement holds for $\overline{\Phi}$. For $0 \leq k \leq m - 1$, we have

$$\begin{aligned}
d_k(\Phi_{i_1\dots i_m}) &= d_k((\Psi \bullet \overline{\Phi})_{i_1\dots i_m}) \\
&= \sum_{s_k} (\Psi \bullet \overline{\Phi})^{s_k}_{i_1\dots i_m} \\
&= \sum_{s_k} \left(\sum_{r_m} \Psi_{i_m,r_m} \otimes \overline{\Phi}^{s_k,r_m}_{i_1\dots i_{m-1}} \right) \\
&= \sum_{r_m} \Psi_{i_m,r_m} \otimes \left(\sum_{s_k} \overline{\Phi}^{s_k,r_m}_{i_1\dots i_{m-1}} \right) \\
&= \sum_{r_m} \Psi_{i_m,r_m} \otimes d_k(\overline{\Phi}_{i_1\dots i_{m-1}})^{r_m} \\
&= (\Psi \bullet d_k \overline{\Phi})_{i_1\dots i_m}
\end{aligned}$$

and for $k = m$ a similar computation gives $d_m(\Phi_{i_1\dots i_m}) = (d_m\Psi \bullet \overline{\Phi})_{i_1\dots i_m}$. Two non-degenerate m-simplices σ, τ of S^m have the same face if

1. $d_m\sigma = d_m\tau$ where $\sigma = (\sigma', \sigma_0)$ and $\tau = (\sigma', \sigma_1)$ with σ' is a non-degenerate $(m-1)$-simplex of S^{m-1}, or

2. $d_k\sigma = d_\ell\tau$ where $\sigma = (\sigma', \sigma_i)$ and $\tau = (\tau', \sigma_i)$ with σ', τ' are non-degenerate $(m-1)$-simplices of S^{m-1} such that $d_k\sigma' = d_\ell\tau'$.

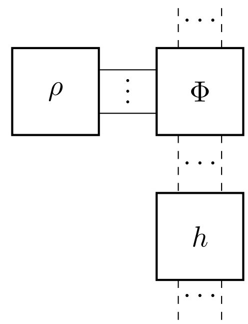
For the first case, we have

$$\begin{aligned}
d_m(\Phi_{i_1\dots i_{m-1}0}) &= \left(\sum_{s_m} \Psi^{s_m}_0 \right) \bullet \overline{\Phi}_{i_1\dots i_{m-1}} \\
&= \sum_{r_m} \left(\sum_{s_m} \Psi^{s_m}_{0,r_m} \right) \otimes \overline{\Phi}^{r_m}_{i_1\dots i_{m-1}} \\
&= \sum_{s_m} \left(\sum_{r_m} \Psi^{s_m}_{0,r_m} \otimes \overline{\Phi}^{r_m}_{i_1\dots i_{m-1}} \right) \\
&= \sum_{s_m} \left(\sum_{r_m} \Psi^{s_m}_{1,r_m\oplus 1} \otimes \overline{\Phi}^{r_m\oplus 1}_{i_1\dots i_{m-1}} \right) \\
&= \left(\sum_{s_m} \Psi^{s_m}_1 \right) \bullet \overline{\Phi}_{i_1\dots i_{m-1}} \\
&= d_m(\Phi_{i_1\dots i_{m-1}1})
\end{aligned}$$

where we used $\Psi_{i,r} = \Psi_{i',r'}$ when $i + r = i' + r' \mod 2$. The second case is similar and follows from our assumption on $\overline{\Phi}$ that the corresponding assignment is simplicial. $\square$

6.3 Computing Boolean functions

We are interested in the computational power of adaptive quantum computation in the vertical direction. That is, given a Boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$ we want to compute f using an adaptive instrument of the form



where

- $\rho \in \mathrm{Den}((\mathbb{C}^2)^{\otimes m})$ is a quantum state regarded as an instrument,
- Φ is an m-Bell instrument (Definition 44),
- h is an affine Boolean instrument with input $\mathbb{Z}_2^m$ and output $\mathbb{Z}_2^\ell$.

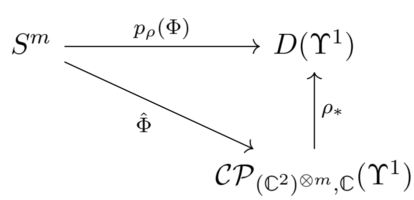
The resulting composition will result in a Kleisli map

$$p(\rho, \Phi, h) : \mathbb{Z}_2^m \longrightarrow D(\mathbb{Z}_2^\ell).$$

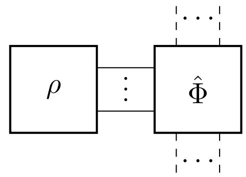
Definition 75. Given a Kleisli map $p : \mathbb{Z}_2^m \to D(\mathbb{Z}_2^\ell)$ and a Boolean function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$, the *average success probability* for computing f with p is defined by

$$p_{\mathrm{succ}}(p, f) = \frac{1}{2^m} \sum_{r \in \mathbb{Z}_2^m} p_r^{f(r)}.$$

The m-Bell instrument Φ can be realized as a simplicial instrument using Construction 73. Then, using the quantum state ρ we can form a simplicial distribution



where the vertical map is given by the instrument:



Our main result relates the success probability, the contextual fraction, and a distance measure between Boolean functions, which we define next.

Definition 76. Given Boolean functions $f, g : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$, the *average distance* between them is defined by

$$d(f,g) = \frac{1}{2^m} |\{r \in \mathbb{Z}_2^m \colon f(r) \neq g(r)\}|.$$

The *average distance of f to the closest affine Boolean function* is defined by

$$\nu(f) = \min\{d(f,h) \colon h : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell \text{ affine Boolean function}\}.$$

Theorem 77. *Let $f : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$ be a Boolean function. Given an m-qubit quantum state ρ, an m-Bell instrument Φ, and an affine Boolean function $h : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$, the average success probability of $p(\rho, \Phi, h)$ computing f satisfies*

$$p_{\text{succ}}\big(p(\rho,\Phi,h), f\big) \;\leq\; 1 - \text{NCF}\left(p_\rho(\Phi)\right)\nu(f).$$

Proof. The proof idea is similar to [17, Theorem 3]. We begin with the decomposition

$$p_\rho(\Phi) = \lambda p + (1-\lambda) q$$

where p is a non-contextual, q is contextual, and λ is the non-contextual fraction of the simplicial distribution $p_\rho(\Phi)$ (Definition 68). The Kleisli map $p(\rho, \Phi, h)$ coincides with the Kleisli composition $h \circ \overline{p_\rho(\Phi)}$. Therefore, we obtain the decomposition

$$p(\rho,\Phi,h) = \lambda(h \circ \bar{p}) + (1-\lambda) h \circ \bar{q}.$$

We have

$$\begin{aligned}
p_{\text{succ}}(p(\rho,\Phi,h), f) &= \frac{1}{2^m} \sum_{r \in \mathbb{Z}_2^m} p(\rho,\Phi,h)_r^{f(r)} \\
&= \frac{1}{2^m} \sum_{r \in \mathbb{Z}_2^m} (\lambda (h \circ \bar{p})_r^{f(r)} + (1-\lambda)(h \circ \bar{q})_r^{f(r)}) \\
&= \lambda \frac{1}{2^m} \sum_{r \in \mathbb{Z}_2^m} (h \circ \bar{p})_r^{f(r)} + (1-\lambda) \frac{1}{2^m} \sum_{r \in \mathbb{Z}_2^m} (h \circ \bar{q})_r^{f(r)} \\
&= \lambda p_{\text{succ}}(h \circ \bar{p}, f) + (1-\lambda) p_{\text{succ}}(h \circ \bar{q}, f) \\
&\leq \lambda p_{\text{succ}}(h \circ \bar{p}, f) + (1-\lambda) \\
&= 1 - \lambda(1 - p_{\text{succ}}(h \circ \bar{p}, f)).
\end{aligned}$$

Writing $p = \sum_i \lambda_i \delta^{s_i}$ we obtain

$$\begin{aligned}
1 - p_{\text{succ}}(h \circ \bar{p}, f) &= 1 - \sum_i \lambda_i p_{\text{succ}}(h \circ \overline{s_i}, f) \\
&= 1 - \frac{1}{2^m} \sum_{r \in \mathbb{Z}_2^m} \delta_{h \circ \overline{s_i}(r), f(r)} \\
&= \frac{1}{2^m} |\{r \in \mathbb{Z}_2^m \colon h \circ \overline{s_i}(r) \neq f(r)\}| \\
&= d(h \circ \overline{s_i}, f) \\
&\geq \nu(f).
\end{aligned}$$

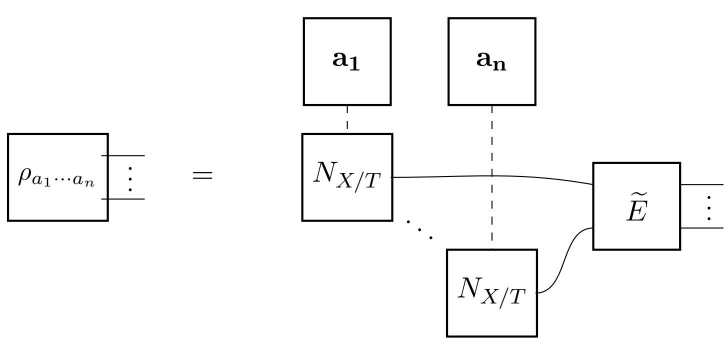
Combining the two inequalities gives the desired result. In the last inequality we used the fact that $h \circ \overline{s_i}$ is affine since $\overline{s_i}$ affine by Proposition 72. □

Remark 78. In our proof, the original argument of [16], which shows that an MBQC running on a non-contextual quantum state can compute only affine Boolean functions, is simplified by a structural property of (double) port graphs—namely, their *acyclicity* (Definition 3). This property naturally induces a time ordering on the application of operations within our double port graphs. We note that this feature plays a crucial role in the proof of Lemma 74.

Corollary 79. *If $p(\rho, \Phi, h) = \delta^f$ for some $f : \mathbb{Z}_2^m \to \mathbb{Z}_2^\ell$ that is not affine then $p_\rho(\Phi)$ is strongly contextual.*

Proof. The condition $p(\rho, \Phi, h) = \delta^f$ implies that the success probability of $p(\rho, \Phi, h)$ computing f is 1. Then, the inequality in Theorem 77 gives that the non-contextual fraction of $p_\rho(\Phi)$ is zero since $\nu(f) \neq 0$ by assumption. □

Using the MBPC labels (Section 5.3) we can prepare a quantum state of the form



where $\mathbf{a_1}, \cdots, \mathbf{a_n}$ are either **0**- or **1**-state labels and $\widetilde{E}$ label consists of only E-labels. A Bell instrument in MBPC consists of $M_{X/Y}$-labels that are composed in an adaptive manner using the affine Boolean labels for the control of the inputs.

Example 80. The OR-gadget of Definition 56, constructed in MBPC, computes a non-affine Boolean function. The quantum state employed in this construction, the GHZ state, is well known to be strongly contextual [37].

Other constructions in measurement-based models are also capable of computing Boolean functions that are not affine; see, for instance, [18, 19].

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A Double categories

Double categories admit composition operations in two directions—horizontal and vertical—which may be either strict or weak. While the strict case is relatively straightforward to define, the weak versions are more subtle. In this paper, we focus on the 1-object case, where the formulation becomes more concrete, and provide an explicit description of this setting.

Definition 81. A *strict double category* D is an internal category in the category Cat of (small) categories.

Let us unravel this definition. The double category D comes with an object category D_0 and a morphism category D_1 together with the source and target functors $s, t : \mathsf{D}_1 \to \mathsf{D}_0$. To restore the hidden symmetry in this structure the convention is to employ the following language:

- *objects*: $\mathrm{Ob}(\mathsf{D}_0)$,
- *vertical morphisms*: $\mathrm{Mor}(\mathsf{D}_0)$,
- *horizontal morphisms*: $\mathrm{Ob}(\mathsf{D}_1)$,
- *squares*: $\mathrm{Mor}(\mathsf{D}_1)$.

A square α of type $\left(u \, {}^{f}_{g} \, v \right)$ is depicted by a diagram of the form

$$\begin{array}{ccc} c & \xrightarrow{f} & d \\ {\scriptstyle u}\big\downarrow & \alpha & \big\downarrow{\scriptstyle v} \\ c' & \xrightarrow[g]{} & d' \end{array}$$

where $c, c'd, d'$ are the objects, f, g are the horizontal morphisms, and u, v are the vertical morphisms. Given another $\left(v \, {}^{h}_{k} \, w \right)$-square β we can form the horizontal composition to obtain

$$\begin{array}{ccc} c & \xrightarrow{h \circ f} & e \\ {\scriptstyle u}\big\downarrow & \beta \circ \alpha & \big\downarrow{\scriptstyle w} \\ c' & \xrightarrow[k \circ g]{} & e' \end{array}$$

On the other hand, we can compose vertically with a $\left(w \, {}^{g}_{h} \, t \right)$-square to obtain

$$\begin{array}{ccc} c & \xrightarrow{f} & d \\ {\scriptstyle w \bullet u}\big\downarrow & \beta \bullet \alpha & \big\downarrow{\scriptstyle t \bullet v} \\ c'' & \xrightarrow[h]{} & d'' \end{array}$$

These compositions satisfy associativity and unitality. The horizontal and vertical units are denoted by

$$\begin{array}{ccc} c & \xrightarrow{\mathrm{Id}_c} & c \\ {\scriptstyle u}\downarrow & \mathrm{Id}^h_u & \downarrow{\scriptstyle u} \\ c' & \xrightarrow[\mathrm{Id}_{c'}]{} & c' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} c & \xrightarrow{f} & d \\ {\scriptstyle 1_c}\downarrow & \mathrm{Id}^v_f & \downarrow{\scriptstyle 1_d} \\ c & \xrightarrow[f]{} & d \end{array}$$

and they satisfy $\mathrm{Id}^h_{1_c} = \mathrm{Id}^v_{\mathrm{Id}_c}$. Finally, given four squares that are compatible as in

$$\begin{array}{ccccc} c & \longrightarrow & d & \longrightarrow & e \\ \downarrow & \alpha & \downarrow & \beta & \downarrow \\ c' & \longrightarrow & d' & \longrightarrow & e' \\ \downarrow & \theta & \downarrow & \gamma & \downarrow \\ c'' & \longrightarrow & d'' & \longrightarrow & e'' \end{array}$$

first composing horizontally and then vertically is the same as first composing horizontally and then vertically. This is called the interchange law which can be written as

$$(\gamma \circ \theta) \bullet (\beta \circ \alpha) = (\gamma \bullet \beta) \circ (\theta \bullet \alpha).$$

We focus on 1-object double categories, for which the underlying data simplifies considerably. The idea parallels the one-dimensional case: a 1-object strict 2-category is equivalent to a strict monoidal category, and likewise, a 1-object bicategory—a weakened form of a strict 2-category in which associativity and unit laws hold only up to coherent isomorphism—corresponds to a monoidal category. For a (strict) monoidal category C, we denote by $B(\mathsf{C})$ its *delooping*, that is, the associated 1-object (strict 2-category) bicategory. Then $B(\mathsf{C})$ has a single object, with

- 1-morphisms given by the objects of C,
- 2-morphisms given by the morphisms of C.

Composition of 1-morphisms is given by the tensor product, while composition of 2-morphisms is the usual composition of morphisms in C.

A 1-object double category can be described equivalently by a pair of strict monoidal categories interacting in a compatible way. Given a 1-object double category D, there are two associated strict monoidal categories. The first, denoted D_H, arises from the horizontal composition of squares: its objects are the vertical morphisms $\mathrm{Ob}(\mathsf{D}_1)$, composition corresponds to vertical square composition, and the tensor product $\otimes_h$ is given by horizontal square composition. Dually, the second monoidal category, denoted D_V, arises from the vertical composition of squares: its objects are the horizontal morphisms $\mathrm{Mor}(\mathsf{D}_0)$, composition corresponds to horizontal square composition, and the tensor product $\otimes_v$ is given by vertical square composition. These two monoidal categories are linked by functors that capture how horizontal and vertical structures interact. Specifically, there are horizontal source and target functors

$$s_H,\, t_H : \mathsf{D}_V \longrightarrow B(\mathsf{D}_H),$$

and vertical source and target functors

$$s_V,\, t_V : \mathsf{D}_H \longrightarrow B(\mathsf{D}_V),$$

where $B(-)$ denotes the delooping construction. The interaction between the two directions is governed by the *interchange law*.

Conversely, starting from two strict monoidal categories $(\mathsf{H}, \otimes_h)$ and $(\mathsf{V}, \otimes_v)$ with the same set of morphisms, equipped with source and target functors

$$s_H, t_H : \mathsf{V} \to B(\mathsf{H}) \quad \text{and} \quad s_V, t_V : \mathsf{H} \to B(\mathsf{V}),$$

and satisfying the interchange law, one can reconstruct a 1-object double category D. In this double category, the horizontal morphisms are given by $\mathrm{Ob}(\mathsf{H})$, the vertical morphisms by $\mathrm{Ob}(\mathsf{V})$, and the squares by the common set of morphisms $\mathrm{Mor}(\mathsf{H}) = \mathrm{Mor}(\mathsf{V})$. Hence, a 1-object double categorSectiony may be viewed equivalently as a pair of mutually compatible strict monoidal categories encoding its horizontal and vertical compositions.

We will also use a weaker version of double categories, in which one of the compositions—horizontal or vertical—is weakened in the sense that associativity and unitality hold only up to coherent isomorphisms [38, 39].

Definition 82. A *pseudo-double category* D is a weakly internal category in the strict 2-category Cat of (small) categories.

In this definition, the horizontal composition is weakened. Using a similar approach, we can describe the data in the 1-object case more explicitly. A 1-*object pseudo-double category* D consists of the following data:

- a monoidal category $(\mathsf{H}, \otimes_h)$ and a strict monoidal category $(\mathsf{V}, \otimes_v)$ sharing the same morphism set,
$$\mathrm{Mor}(\mathsf{H}) = \mathrm{Mor}(\mathsf{V}),$$
and satisfying the interchange law
$$(\gamma \otimes_h \theta) \otimes_v (\beta \otimes_h \alpha) = (\gamma \otimes_v \beta) \otimes_h (\theta \otimes_v \alpha);$$
- horizontal source and target pseudo-functors
$$s_H,\, t_H : \mathsf{V} \longrightarrow B(\mathsf{H}),$$
and vertical source and target functors
$$s_V,\, t_V : \mathsf{H} \longrightarrow B(\mathsf{V}).$$

Definition 83. For a 1-object pseudo-double category D, we define the following associated monoidal categories:

- The *horizontal monoidal category* $H(\mathsf{D})$, whose objects are $\mathrm{Mor}(\mathsf{D}_0)$ and whose morphisms $u \to v$ are given by the horizontal composition of squares of the form

$$\left(u \,\substack{\mathrm{Id}_c \\ \\ \mathrm{Id}_d}\, v \right),$$

  with tensor product defined by the vertical composition of such squares.

- The *vertical (strict) monoidal category* $V(\mathsf{D})$, whose objects are $\mathrm{Ob}(\mathsf{D}_1)$ and whose morphisms $f \to g$ are given by the vertical composition of squares of the form

$$\left(1_c \,\substack{f \\ \\ g}\, 1_d \right),$$

  with tensor product defined by the horizontal composition of such squares.

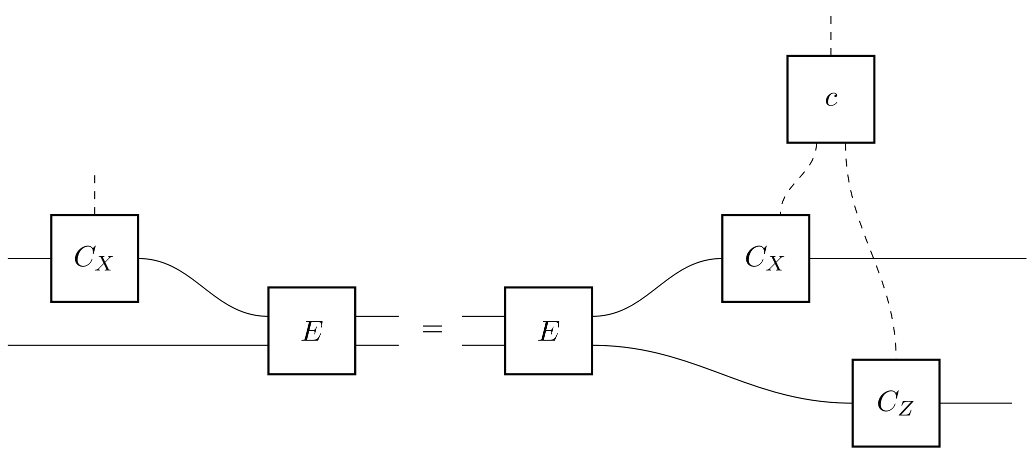
Since our focus is on 1-object double categories, we work with the horizontal and vertical *monoidal* categories rather than the corresponding bicategories, which are typically the more general structures. The deloopings of these monoidal categories recover the associated horizontal and vertical bicategories.

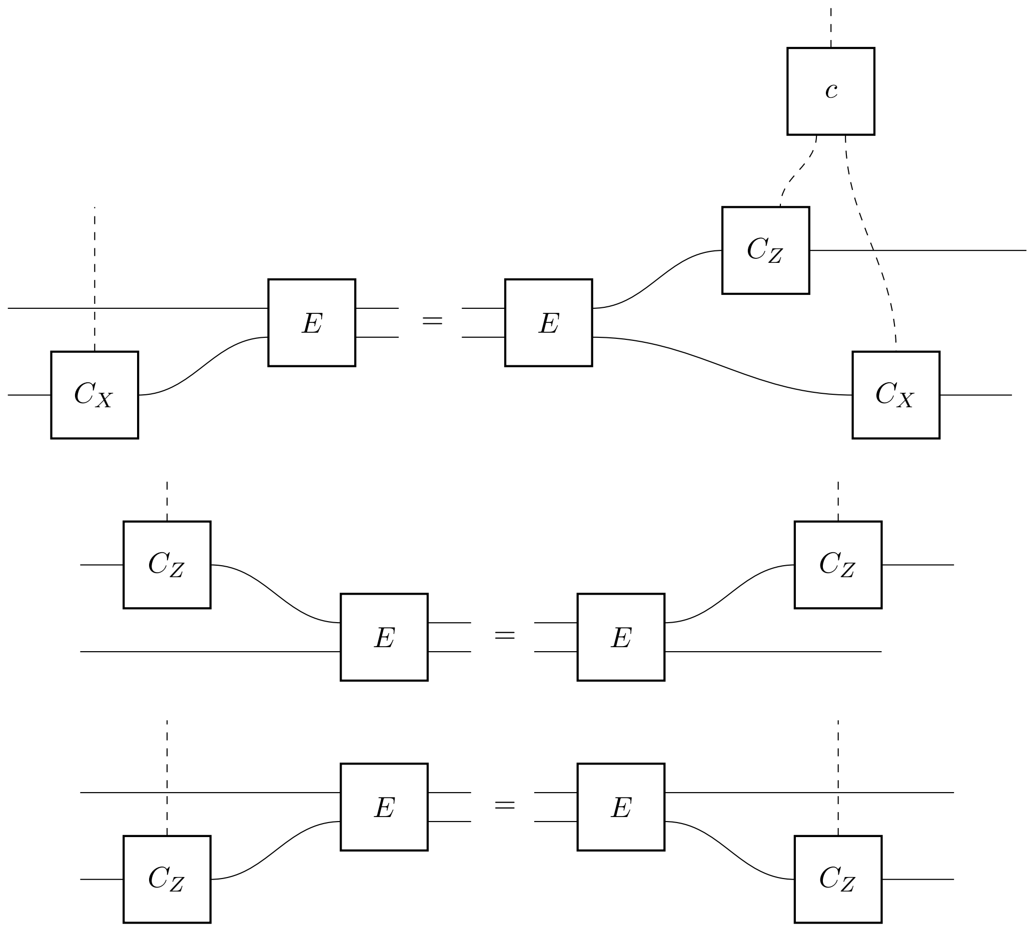
In fact, it is possible to weaken the composition in both directions. The resulting structure is known as a *doubly weak double category*, as introduced in [23]. This notion generalizes and encompasses other weakened forms of double categories, such as those described in [40, 41].

B The standard form

Following [27], we introduce the following *rewrite rules* to capture the relations in the image of $\phi_{\mathcal{M}}$, that is, among the instruments corresponding to the MBQC labels. These rules will subsequently be used to define a quotient double category of $\mathsf{DPG}_{\mathcal{M}}$.

- *Correction vs. entangling rewrite rules*:





It is straightforward to show the correctness of these rewrite rules. We demonstrate this explicitly for the first one. The others follow similarly. The left-hand side corresponds to the instrument $\Phi_{EC} := \Phi_E \circ \Phi_{C_X}$. For $r \in \mathbb{Z}_2$ we have that $(\Phi_{EC})_r \in \mathrm{CP}((\mathbb{C}^2)^{\otimes 2}, (\mathbb{C}^2)^{\otimes 2})$ given by

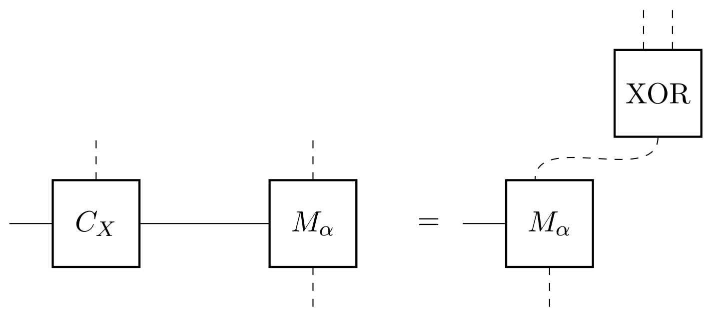
$$\begin{aligned}(\Phi_{EC})_r(-) &= E(X^r \otimes I)(-)(X^r \otimes I)E \\ &= (X^r \otimes Z^r)E(-)E(X^r \otimes Z^r),\end{aligned}$$

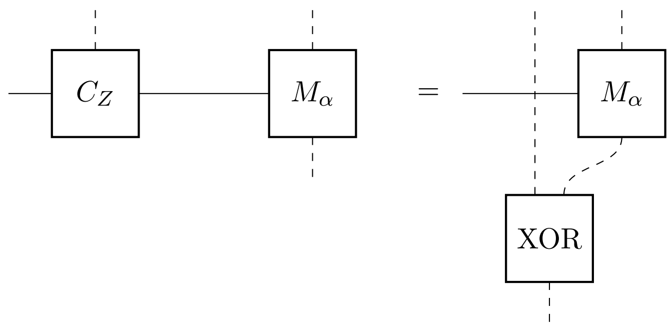
where the second equality follows from the conjugation action of E. Consider now the right-hand side of the diagram with corresponding instrument $\Phi_{CE} := ((\Phi_{C_X} \otimes \Phi_{C_z}) \bullet \Phi_c) \circ \Phi_E$. Evaluated on $r \in \mathbb{Z}_2$ we have

$$\begin{aligned}\Phi_{CE}(r)(-) &= \sum_{s,t} \delta_{r,s}\delta_{r,t}(X^s \otimes Z^t)E(-)E(X^s \otimes Z^t) \\ &= (X^r \otimes Z^r)E(-)E(X^r \otimes Z^r),\end{aligned}$$

where the sum over $(s,t) \in \mathbb{Z}_2^2$ comes from the vertical composition of the Boolean instrument Φ_c with $\Phi_{C_X} \otimes \Phi_{C_Z}$. This is precisely the instrument Φ_{EC}.

- *Correction vs. measurement rewrite rules*:





We now prove the validity of these rewrite rules. Let us begin with the first and introduce the instrument $\Phi_{CM} := \Phi_{M_\alpha} \circ \Phi_{C_X}$. For $((r,s),t) \in \mathbb{Z}_2^2 \times \mathbb{Z}_2$ we have that

$$\begin{aligned}
(\Phi_{CM})^t_{r,s}(-) &= \mathrm{Tr}\left(\Pi^t_{(-1)^s\alpha} X^r(-)X^r \Pi^t_{(-1)^s\alpha}\right) \\
&= \mathrm{Tr}\left(X^r \Pi^t_{(-1)^{r+s}\alpha}(-)\Pi^t_{(-1)^{r+s}\alpha} X^r\right) \\
&= \mathrm{Tr}\left(\Pi^t_{(-1)^{r+s}\alpha}(-)\Pi^t_{(-1)^{r+s}\alpha}\right) \\
&= (\Phi_{M_\alpha})^t_{r+s}(-),
\end{aligned}$$

where in the second equality we used that $X^r(\cos\alpha\, X + \sin\alpha\, Y)X^r = \cos\alpha\, X + (-1)^r \sin\alpha\, Y = \cos\alpha\, X + \sin(-1)^r\alpha\, Y$ and then used the cyclic property of the trace in the third equality. Alternatively we also have the instrument $\Phi_{M\,\mathrm{X}} := \Phi_{M_\alpha} \bullet \Phi_{\mathrm{XOR}}$. Let $((r,s),t) \in \mathbb{Z}_2^2 \times \mathbb{Z}_2$ then we have

$$\begin{aligned}
(\Phi_{M\,\mathrm{X}})^t_{r,s}(-) &= \sum_u \delta_{s\oplus r,u} \mathrm{Tr}\left(\Pi^t_{(-1)^u\alpha}(-)\Pi^t_{(-1)^u\alpha}\right) \\
&= \mathrm{Tr}\left(\Pi^t_{(-1)^{r+s}}(-)\Pi^t_{(-1)^{r+s}}\right) = (\Phi_{M_\alpha})^t_{r+s}(-).
\end{aligned}$$

For the second rewrite rule let us define the maps $\Phi_{CM'} := \Phi_{M_\alpha} \circ \Phi_{C_Z}$ and $\Phi_{\mathrm{X}M} := \Phi_{\mathrm{XOR}} \bullet \Phi_{M_\alpha}$. Computing the first map for $((r,s),t) \in \mathbb{Z}_2^2 \times \mathbb{Z}_2$ gives

$$
\begin{aligned}
(\Phi_{CM})^t_{r,s}(-) &= \mathrm{Tr}\left(\Pi^t_{(-1)^s\alpha} Z^r (-) Z^r \Pi^t_{(-1)^s\alpha}\right) \\
&= \mathrm{Tr}\left(Z^r \Pi^{r+t}_{(-1)^s\alpha}(-)\Pi^{r+t}_{(-1)^s\alpha} Z^r\right) \\
&= \mathrm{Tr}\left(\Pi^{r+t}_{(-1)^s\alpha}(-)\Pi^{r+t}_{(-1)^s\alpha}\right) \\
&= (\Phi_{M_\alpha})^{r+t}_s(-),
\end{aligned}
$$

where in the second equality we used that $Z^r X = (-1)^r X Z^r$ and $Z^r Y = (-1)^r Y Z^r$ since Z anticommutes with X and Y. On the other hand, we also have that

$$
\begin{aligned}
(\Phi_{M\mathrm{X}})^t_{r,s}(-) &= \sum_u \delta_{r\oplus t,u} \mathrm{Tr}\left(\Pi^u_{(-1)^s\alpha}(-)\Pi^u_{(-1)^s\alpha}\right) \\
&= \mathrm{Tr}\left(\Pi^{r+t}_{(-1)^s}(-)\Pi^{r+t}_{(-1)^s}\right) = (\Phi_{M_\alpha})^{r+t}_s(-).
\end{aligned}
$$

For the standard form we introduce the following label sets

- *preparation label set* $\mathcal{M}_P$ consists of the N_X-label and E-label.
- *measurement label set* $\mathcal{M}_M$ is the union of M_α where $\alpha = 0, \pm\pi/4$, and $\mathcal{B}_\oplus$,
- *correction label set* $\mathcal{M}_C$ consists of C where $C = X, Z$.

Proposition 84. *Let $\overline{\mathsf{DPG}}_\mathcal{M}$ denote the quotient double category obtained by imposing the rewrite rules above. Then, in this category any $\mathcal{M}$-labeled double port graph Γ can be represented by a unique standard form $\Gamma_C \circ \Gamma_E \circ \Gamma_P$ where Γ_A is a $\mathcal{M}_A$-labeled double port graph with $A = P, M, C$.*

Proof. Proof is the same as [27, Theorem 3]. □