Possibilistic empirical models and simplicial distributions

Aziz Kharoof* and Cihan Okay†

Department of Mathematics, Bilkent University, Ankara, Turkey

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Quantum contextuality [1,2] is a fundamental feature of quantum theory, capturing the impossibility of describing quantum measurement statistics using local hidden-variable models. A natural way to represent the resulting joint probability distributions is through sheaf theory, an approach due to Abramsky and Brandenburger [3] based on empirical models. More recently, a simplicial approach [4] has been introduced based on the theory of simplicial distributions, emphasizing topological and categorical structures [5,6]. In this work, available at [7], we characterize possibilistic models within both frameworks as sub-scenarios and use this characterization to develop categorical and simplicial criteria for detecting extremal contextual distributions.

In the sheaf-theoretic setting, the key idea is to organize measurements into a simplicial complex Σ and to combine measurement outcomes and their compatibility relations into a presheaf. Our approach is more general, in that we start from a functor

$$F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

on the poset category of simplices $\mathbf{C}_\Sigma$. We only require that this functor satisfies $F(\sigma) \neq \varnothing$ for every simplex $\sigma \in \Sigma$, and call such a functor an *event scenario*. The category of event scenarios is denoted by **eScen**; it contains the category of scenarios in the sense of sheaf theory [8]. They were initially introduced in [9] to develop a suitable category for studying resource theory of contextuality. The probabilistic data associated with collections of jointly performable measurements are encoded by an *empirical model* p on an event scenario F, which assigns to each context $\sigma \in \Sigma$ a probability distribution p_σ on $F(\sigma)$, subject to no-signaling conditions: whenever $\tau \subseteq \sigma$, the marginal of p_σ along the restriction map $F(\sigma) \to F(\tau)$ coincides with p_τ. We denote by $\mathsf{Emp}(F)$ the set of all empirical models on F.

An event scenario F is called a *proper event scenario* if, in addition, for every face inclusion $\sigma \hookrightarrow \tau$ in $\mathbf{C}_\Sigma$, the induced map

$$F(\tau) \longrightarrow F(\sigma)$$

is surjective. We define the functor of sub-proper event scenarios $\mathsf{ESub}\colon \mathbf{eScen} \to \mathbf{Set}$ using a relative version of the Grothendieck construction that retains 2-categorical data and is central to our framework.

Theorem 0.1. *The functors*

$$\mathsf{Emp}_{\mathbb{B}}\colon \mathbf{eScen} \longrightarrow \mathbf{Set} \qquad \text{and} \qquad \mathsf{ESub}\colon \mathbf{eScen} \longrightarrow \mathbf{Set}$$

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[*] aziz.kharoof@bilkent.edu.tr

[†] cihan.okay@bilkent.edu.tr

are equivalent.

This theorem identifies possibilistic empirical models with sub-proper event scenarios, providing a purely categorical description of the support of empirical models. Combined with the fact that extremality is determined by the possibilistic collapse, this leads to a characterization of extremal empirical models; see [7, Theorem 4.10].

A more general viewpoint arises from the theory of simplicial sets. In this approach, measurement scenarios are represented by surjective simplicial set maps, called *bundle scenarios* [10],

$$f\colon E \to X.$$

The category of bundle scenarios is denoted by $\mathbf{sScen}$. The probabilistic data are encoded by simplicial distributions as given in [7, Definition 2.22]. We denote by $\mathsf{sDist}(f)$ the set of all simplicial distributions on f. A *proper bundle scenario* is required to satisfy an additional lifting condition (see [7, Definition 2.17]). We also define the functor of sub-proper bundle scenarios $\mathsf{Sub}\colon \mathbf{sScen} \longrightarrow \mathbf{Set}$ again using the relative Grothendieck construction and prove the following result.

Theorem 0.2. *The functors*

$$\mathsf{sDist}_{\mathbb{B}}\colon \mathbf{sScen} \longrightarrow \mathbf{Set} \qquad \textit{and} \qquad \mathsf{Sub}\colon \mathbf{sScen} \longrightarrow \mathbf{Set}$$

are equivalent.

Theorem 0.2 can be seen as a geometric counterpart of Theorem 0.1. The bundle approach makes precise and extends earlier attempts to study contextuality via bundle diagrams [12, 13]. Using these equivalences, we derive sufficient conditions for extremality of contextual distributions. Below, we present the version for simplicial distributions; the counterpart for empirical models can be found in [7].

Let $f\colon E \to X$ be a bundle scenario. We say that two simplices $e, \tilde{e} \in E$ are *strongly connected by f* if there exists a finite sequence of simplices $e = e_1, e_2, \ldots, e_m = \tilde{e}$ such that for each consecutive pair e_i, e_{i+1} there exist ordinal maps θ_1, θ_2 with $\theta_1^*(e_i) = \theta_2^*(e_{i+1})$, and this common restriction uniquely lifts along f to the simplices e_i and e_{i+1}. This notion is motivated by the underlying geometry of n-partite Bell scenarios. In the simplicial framework, the corresponding measurement space is given by the n-fold join of the two-point set. This yields a particular triangulation of the n-dimensional sphere satisfying our strong connectivity condition.

Theorem 0.3. *Let p be a simplicial distribution on a bundle scenario $f\colon E \to X$. Let $g\colon E' \to X$ be the proper bundle scenario associated with the Boolean projection of p. If every pair of generator simplices of E' is strongly connected by g, then p is a vertex of $\mathsf{sDist}(f)$.*

We apply these conditions to detect contextual vertices in concrete classes of measurement scenarios. In particular, when the measurement simplicial complex is the boundary of the n-simplex (an important scenario that plays a key role in Vorobev's theorem [16]), we provide an effective method for identifying certain contextual vertices directly from the associated sub-scenarios, without requiring full facet or vertex enumeration. In addition, we use this method to explain a known class of vertices in the $(n, 2, 2)$ Bell scenario, i.e., n parties with two measurements per party and two outcomes per measurement. Finally, we apply Theorem 0.3 to clarify the geometric structure underlying extremal contextual simplicial distributions in the $(3, 2, 2)$ Bell scenario capturing examples described in [17].

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Aziz Kharoof* and Cihan Okay†

Department of Mathematics, Bilkent University, Ankara, Turkey

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Abstract

We study joint probabilities using two complementary mathematical frameworks: the sheaf-theoretic formulation of empirical models and the simplicial approach based on simplicial bundle scenarios and simplicial distributions. We use additional structural conditions defining *proper event scenarios* and *proper simplicial bundle scenarios*, and show that Possibilistic empirical models and possibilistic simplicial distributions are completely characterized by sub-proper event scenarios and sub-proper bundle scenarios, respectively. This yields equivalences between the corresponding functors, providing a purely categorical and topological descriptions of possibilistic joint probabilities. Using these equivalences, together with the fact that extremality of probabilistic models is determined by their Boolean projections, we derive sufficient conditions for extremality of empirical models and simplicial distributions. In the sheaf-theoretic setting, we obtain a vertex criterion expressed in terms of connectivity and minimality properties of the associated sub-proper event scenarios. In the simplicial setting, we formulate a geometric criterion for extremality based on lifting properties and simplicial connectivity of the induced sub-proper bundle scenarios. We apply these results to concrete classes of measurement scenarios, where our criteria enable the detection of contextual vertices without requiring full polytope enumeration. This approach provides geometric insight into extremal contextual models and clarifies structural features of the associated contextual polytopes.

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*aziz.kharoof@bilkent.edu.tr

†cihan.okay@bilkent.edu.tr

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1 Introduction

Contextuality is a fundamental feature of probabilistic models arising in quantum theory and related areas, capturing the impossibility of consistently extending locally defined probability assignments to a global model. Over the past decade, several mathematical frameworks have been developed to formalize and study contextuality, most notably the sheaf-theoretic approach of Abramsky and Brandenburger [1] and, more recently, the simplicial distribution apprach [2] that emphasize topolgical and categorical structure [3, 4].

In the sheaf-theoretic setting, contextual models are described by *empirical models*, which encode probabilistic data associated with collections of measurements that can be jointly performed. The key idea is to organize measurements into a simplicial complex Σ, and to combine measurements, outcomes, and their compatibility relations into a presheaf, called an *event scenario*,

$$F\colon \mathbf{C}_{\Sigma}^{\mathrm{op}} \to \mathbf{Set}.$$

An *empirical model* p on F assigns to each context $\sigma \in \Sigma$ a probability distribution p_σ on the set of events $F(\sigma)$, subject to compatibility conditions expressing no-signalling: whenever $\tau \subseteq \sigma$, the marginal of p_σ along the restriction map $F(\sigma) \to F(\tau)$ coincides with p_τ. We denote by $\mathsf{Emp}(F)$ the set of all empirical models on F. Contextuality is then characterized by the failure of an empirical model to arise as the family of marginals of a single global distribution on the maximal contexts. This perspective has proven highly effective in unifying classical, quantum, and more general non-classical models within a single mathematical framework. Moreover, it enables a systematic study of the structure of the associated contextual polytopes [5, 6].

A more generalized viewpoint arises from the theory of simplicial sets. In this approach, measurement scenarios are represented by simplicial set maps, called *simplicial bundle scenarios* [7],

$$f\colon E \to X$$

The probabilistic data are encoded by simplicial distributions, A *simplicial distribution* on f is a simplicial set map $p\colon X \to D(E)$ that makes the following diagram commutes:

$$\begin{tikzcd} & D(E) \arrow[d, "D(f)"] \\ X \arrow[ur, "p"] \arrow[r, "\delta_X"'] & D(X), \end{tikzcd}$$

where D is the distribution monad [8]. On other words, p assigns to each simplex $x \in X_n$ a probability distribution on the finite fiber $f_n^{-1}(x)$, subject to compatibility conditions with respect to the face and degeneracy maps of the simplicial sets E and X. We denote by $\mathsf{sDist}(f)$ the set of all simplicial distributions on f. Contextual features of the model are reflected in the failure of such distributions to be determined by a single global assignment compatible with the simplicial structure.

The simplicial distribution approach allows contextuality to be studied using tools from algebraic topology and category theory. In particular, simplicial distributions provide a natural language for describing contextual polytopes and their extremal structure, as developed in [9, 10].

In this work, a *proper event scenario* (Definition 2.4) satisfies additional categorical properties (as in [11]), which make it behaves analogously to a sheaf of events in the sense of [1]. This idea introduced first in [12]. In parallel, we define a *proper simplicial bundle scenario* (Definition 2.17), endowed with additional geometric structure. We show that possibilistic (Boolean-valued) empirical models and possibilistic simplicial distributions correspond precisely to proper event scenarios and proper bundle scenarios, respectively.

Theorem 1.1. *The functors*

$$\mathsf{Emp}_{\mathbb{B}} \colon \mathbf{eScen} \longrightarrow \mathbf{Set} \qquad \text{and} \qquad \mathsf{ESub} \colon \mathbf{eScen} \longrightarrow \mathbf{Set}$$

are equivalent.

This theorem shows that possibilistic empirical models on event scenarios are completely characterized by sub-proper event scenarios, providing a purely combinatorial description of possibilistic emprical models.

Theorem 1.2. *The functors*

$$\mathsf{sDist}_{\mathbb{B}} \colon \mathbf{sScen} \longrightarrow \mathbf{Set} \qquad \text{and} \qquad \mathsf{Sub} \colon \mathbf{sScen} \longrightarrow \mathbf{Set}$$

are equivalent.

This result establishes a direct correspondence between possibilistic simplicial distributions and sub-proper bundle scenarios, allowing possibilistic information to be studied using simplicial and geometric methods (see [12, 13]).

Using these equivalences, together with the fact that extremality is determined by projection to the Boolean semiring,

$$\kappa_F \colon \mathsf{Emp}(F) \longrightarrow \mathsf{Emp}_{\mathbb{B}}(F), \qquad \kappa_f \colon \mathsf{sDist}(f) \longrightarrow \mathsf{sDist}_{\mathbb{B}}(f),$$

we derive sufficient conditions for extremality of contextual models.

Theorem 1.3. *Let p be an empirical model on an event scenario $F \colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$. Let $F' \in \mathsf{ESub}(F)$ be the proper event scenario corresponding to the possibilistic empirical model $\kappa_F(p)$. Assume the following:*

1. *For every pair of maximal simplices $\sigma_1, \sigma_2 \in \Sigma$, there exists a zigzag in $\mathbf{C}_\Sigma$*

$$\sigma_1 \xleftarrow{s_1} \tau_1 \xrightarrow{s_2} \tau_2 \xleftarrow{s_3} \dots \xrightarrow{s_n} \sigma_2, \tag{1}$$

such that $F'(s_i)$ is an isomorphism for every morphism s_i appearing in (1).

2. For any $G \in \mathsf{ESub}(F)$ with $G \leq F'$, there exists a maximal simplex $\sigma \in \Sigma$ such that

$$G(\sigma) = F'(\sigma).$$

Then p is a vertex of the polytope $\mathsf{Emp}(F)$.

A similar result holds in the simplicial setting, providing a sufficient geometric condition for a simplicial distribution to be extremal.

Theorem 1.4. *Let $f\colon E \to X$ be a simplicial bundle scenario. Let $p \in \mathsf{sDist}(f)$ be a simplicial distribution, and let $g\colon E' \to X$ be the proper bundle scenario corresponding to the possibilistic simplicial distribution $\kappa_f(p)$. Assume that for every generating simplex $\sigma \in X_n$ and every commutative diagram*

$$\begin{tikzcd}
\Delta[n-1] \arrow[r] \arrow[d, hook] & E' \arrow[d, "g"] \\
\Delta[n] \arrow[r, "\sigma"] & X
\end{tikzcd}$$

there exists a unique lift. Then p is a vertex of $\mathsf{sDist}(f)$.

We apply the above theorems to detect contextual vertices in concrete classes of measurement scenarios by translating combinatorial and geometric connectivity conditions into explicit criteria for extremality. In particular, when the measurement simplicial complex is the boundary of the n-simplex, Theorem 1.3 provides an effective method for identifying some contextual vertices directly from the associated sub-scenarios, without requiring a full facet or vertex enumeration (see Proposition 4.14). In addition, we use the method to explain a known class of vertices on the $(n, 2, 2)$ Bell scenario (see Prposition 4.18). Meanwhile, we use Theorem **??** for clarifying the geometric structure underlying known extremal contextual simplicial distributions (Example 6.6). This perspective also helps us identify new extremal contextual simplicial distributions (Example 6.7).

Throughout the paper, we work over two semirings:

- $R = \mathbb{R}_{\geq 0}$, the semiring of positive reals.
- $R = \mathbb{B}$, the Boolean semiring with the elements 0 and 1.

The paper is organized as follows. In Section 2, we introduce the categories of event scenarios and simplicial bundle scenarios, together with the corresponding functors of empirical models and simplicial distributions. We also give the tools needed to study extremal contextual models. In Section 3, we define the functor of sub-proper event scenarios and show its equivalence with the functor of possibilistic empirical models. In Section 4, we use the characterization established in the previous section to obtain our main result on extremal empirical models. As an application, we identify contextual vertices in the case where the measurement simplicial complex is the boundary of the n-simplex. In Section 5, we characterize possibilistic simplicial distributions as sub-proper bundle scenarios. This characterization is then used in Section 6 to obtain geometric criteria for certain classes of extremal simplicial distributions. We conclude the paper by illustrating the geometric shape vertices that appear in the literature.

2 Contextual models

In this section, we review two main approaches to the study of joint probabilities and contextuality:

1. Event scenarios [11]: a generalization of the sheaf-theoretic approach, formulated in terms of empirical models [1].
2. Simplicial bundle scenarios [7]: a generalization of the simplicial-set approach, based on simplicial distributions [2].

We later study extremal empirical models and extremal simplicial distributions within these frameworks.

2.1 Event scenarios

We define the category of event scenarios [11, Section 2.1], which contains the category of scenarios in the sense of [14].

Let **Comp** denote the *category of simplicial complexes*, where:.

- Objects are finite (abstract) simplicial complexes (V, Σ), where V is a finite vertex set and $\Sigma \subseteq \mathcal{P}(V)$ is a family of nonempty subsets closed under inclusion; that is, if $\sigma \in \Sigma$ and $\tau \subseteq \sigma$, then $\tau \in \Sigma$. Usually, we write simply Σ instead of (V, Σ), and denote its vertex set by $V(\Sigma)$.
- Morphisms called simplicial maps $f\colon \Sigma_1 \to \Sigma_2$, which is is a function $f\colon V(\Sigma_1) \to V(\Sigma_2)$ such that for every simplex $\sigma \in \Sigma_1$, we have $f(\sigma) = \{f(v) \mid v \in \Sigma_1\} \in \Sigma_2$

Composition and identities are given by the usual composition and identity of functions on vertex sets.

Example 2.1. The *standard n-simplex* Δ^n is the simplicial complex whose vertices are the elements of the set $\{0, 1, \ldots, n\}$, and whose simplices are all nonempty subsets of $\{0, 1, \ldots, n\}$. That is,

$$\Delta^n = \{\, \sigma \subseteq \{0, 1, \ldots, n\} \mid \sigma \neq \emptyset \,\}.$$

The *boundary* of Δ^n, denoted $\partial\Delta^n$, is the subcomplex consisting of all proper faces of Δ^n, that is,

$$\partial\Delta^n = \Delta^n - \{\{0, \ldots, n\}\}.$$

Here is another example of a family of simplicial complexes that we will use later to represent Bell scenarios.

Definition 2.2. For natural numbers n and m, let $A_1, \ldots, A_n$ be sets satisfying:

- $|A_i| = m$ for every $1 \leq i \leq n$.
- $A_i \cap A_j = \emptyset$ for every $1 \leq i < j \leq n$.

We define $B(n, m)$ to be the following simplicial complex:

$$\{\{x_1, \ldots, x_k\} : 1 \leq i_1 < \cdots < i_k \leq n \ \text{ and } \ x_1 \in A_{i_1}, \ldots, x_k \in A_{i_k}\}$$

Of course, the vertex set here is $\sqcup_{i=1}^{n} A_i$.

Definition 2.3. For a simplicial complex Σ, the *nerve complex* $\hat{N}\Sigma$ is the simplicial complex defined as follows:

- The vertices are the simplices of Σ, i.e., $V(\hat{N}\Sigma) = \Sigma$,
- The simplices of $\hat{N}\Sigma$ are the collections $\{\sigma_1, \cdots, \sigma_k\}$ such that the union $\cup_{i=1}^k \sigma_i$ belongs to Σ.

The nerve complex construction gives a functor

$$\hat{N} \colon \mathbf{Comp} \to \mathbf{Comp}$$

This functor turns out to be a *monad*.

- The *unit* of the monad, $\delta_\Sigma \colon \Sigma \to \hat{N}\Sigma$, is defined by sending a vertex x to the simplex $\{x\}$.
- The *multiplication*, $\mu_\Sigma \colon \hat{N}^2\Sigma \to \hat{N}\Sigma$, sends a vertex $\{\sigma_1, \ldots, \sigma_n\}$ to the union $\bigcup_{i=1}^n \sigma_i$.

A simplicial complex map $\pi \colon \Sigma' \to \hat{N}\Sigma$ corresponds to a *simplicial relation* from Σ' to Σ. In fact, the category **Rel** of simplicial complexes and simplicial relations is equivalent to the Kleisli category $\mathbf{Comp}_{\hat{N}}$. We will denote the Kleisli composition is written as $\diamond$. From a functorial perspective, a simplicial complex map π induces a functor:

$$\overline{\pi} \colon \mathbf{C}_{\Sigma'} \to \mathbf{C}_\Sigma.$$

This functor defined by mapping σ to $\cup_{\tau \in \pi(\sigma)} \tau$, and it satisfies:

We have

$$\overline{\delta_\Sigma} = \mathrm{id}_{\mathbf{C}_\Sigma}, \quad \overline{\pi_2 \diamond \pi_1} = \overline{\pi_2} \circ \overline{\pi_1} \tag{2}$$

For simplicity of notation, the opposite of $\bar{\pi}$ will also be denoted by the same symbol.

Definition 2.4. Let Σ be a simplicial complex. A functor $F \colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$ is called an *event scenario* if it satisfies the following condition:

- **Non-triviality:** for every simplex $\sigma \in \Sigma$, the set $F(\sigma)$ is non-empty.

An event scenario F is called a *proper event scenario* if, in addition, it satisfies:

- **Local surjectivity:** for every face inclusion $\sigma \hookrightarrow \tau$ in $\mathbf{C}_\Sigma$, the induced map

  $$F(\tau) \longrightarrow F(\sigma)$$

  is surjective.

We denote by $\mathsf{Event}(\Sigma)$ the full subcategory of $\mathsf{Fun}(\mathbf{C}_\Sigma^{\mathrm{op}}, \mathbf{Set})$ consisting of event scenarios.

Example 2.5. Given a measurement scenario $T = (\Sigma, O)$ in the sense of [1], where Σ is a simplicial complex and $O = \{O_x\}_{x \in V(\Sigma)}$ is a family of outcome sets, one for each measurement. In the special case where $O_x = \mathbb{Z}_m$ for every $x \in V(\Sigma)$, we write $O = O_{\mathbb{Z}_m}$. The *sheaf of events*

$$\mathcal{E}_T \colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$$

defined by sending σ to $\prod_{x \in \sigma} O_x$ for each simplex $\sigma \in \Sigma$, is an example of a proper event scenario.

Remark 2.6. The notion of a proper event scenario can be seen as a structural generalization of empirical models in the sense of [12, Definition 1]. In contrast to that definition, we do not require the third condition, nor do we assume that the functor is a subpresheaf of the sheaf of events.

Now, we can consider the functor

$$\mathsf{Event}\colon \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Cat} \tag{3}$$

which sends a simplicial complex Σ to the category $\mathsf{Event}(\Sigma)$, and sends a simplicial complex map $\pi\colon \Sigma' \to \hat{N}\Sigma$ to the functor $\pi^*\colon \mathsf{Event}(\Sigma) \to \mathsf{Event}(\Sigma')$, defined by mapping F to $F \circ \overline{\pi}$ and $\alpha\colon F \to G$ to the horizontal composition $\mathrm{id}_{\overline{\pi}} \star \alpha$.

Definition 2.7. We define the *category of event scenarios*, denoted by **eScen**, as the Grothendieck construction (Definition A.1) of the functor in (3):

$$\mathbf{eScen} = \int_{\mathbf{Rel}^{\mathrm{op}}} \mathsf{Event}.$$

For more details on the category **eScen**, see [11].

2.2 Empirical models on event scenarios

We define the functor of empirical models on event scenarios using the relative Grothendieck construction (see Section A), following the approach of [11, Section 2.3].

Definition 2.8. The *distribution monad* (see [15, Section VI]) is a functor $D_R\colon \mathbf{Set} \to \mathbf{Set}$ defined as follows:

- For a set X, the set $D_R(X)$ of distributions on X is defined by

$$D_R(X) = \{P\colon X \to R \mid \ |\{x \in X :\ P(x) \neq 0\}| < \infty \ \text{ and } \sum_{x \in X} P(x) = 1\}.$$

- For a map $f\colon X \to Y$, the map $D_R(f)\colon D_R(X) \to D_R(Y)$ is given by

$$D_R(f)(P)(y) = \sum_{x \in f^{-1}(y)} P(x).$$

The unit of the possibilistic monad, $\delta_X\colon X \hookrightarrow D_R(X)$, sends each $x \in X$ to the *delta distribution* δ^x, defined by

$$\delta^x(x') = \begin{cases} 1, & x' = x, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

When $R = \mathbb{R}_{\geq 0}$ we write D instead of $D_{\mathbb{R}_{\geq 0}}$, and when $R = \mathbb{B}$ we call the monad $D_{\mathbb{B}}$ the *possibilistic monad.* Given $P \in D(X)$ and $x \in X$, we will usually write P^x instead of $P(x)$.

We have a natural projection

$$\kappa_X\colon D(X) \to D_{\mathbb{B}}(X) \tag{5}$$

For any functor $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$, composing with the distribution monad D_R produces a functor $D_R \circ F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$. We define $\mathsf{Emp}_{R,\Sigma}$ to be the composite functor

$$\mathsf{Event}(\Sigma) \xrightarrow{D_R \circ -} \mathsf{Fun}(\mathbf{C}_\Sigma^{\mathrm{op}}, \mathbf{Set}) \xrightarrow{\lim} \mathbf{Set}.$$

Moreover, given a morphism $\pi\colon \Sigma' \to \hat{N}\Sigma$ in **Rel**, for every inclusion $s\colon \sigma \hookrightarrow \tau$ in $\mathbf{C}_{\Sigma'}$, the following diagram—with two arrows of the canonical maps of the limit—commutes:

$$\begin{tikzcd}
 & D_R \circ F(\bar{\pi}(\tau)) \arrow[dd, "D_R \circ F(\bar{\pi}(s))"] \\
\lim(D_R \circ F) \arrow[ur] \arrow[dr] & \\
 & D_R \circ F(\bar{\pi}(\sigma))
\end{tikzcd}$$

Hence, we obtain a natural map

$$\mathsf{Emp}_{R,\pi}\colon \lim(D_R \circ F) \longrightarrow \lim(D_R \circ F \circ \bar{\pi}).$$

Therefore, we have a morphism

$$(\pi^*, \mathsf{Emp}_{R,\pi})\colon \mathbf{eScen} \to \mathbf{Set}$$

in the thick slice category $\mathbf{Cat} /\!\!/ \mathbf{Set}$ (see Definition A.2), as depicted in the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Event}(\Sigma) \arrow[rr, "\pi^*"] \arrow[dr, "\mathsf{Emp}_{R,\Sigma}"'] & \arrow[d, phantom, "\overset{\mathsf{Emp}_{R,\pi}}{\Longrightarrow}"] & \mathsf{Event}(\Sigma') \arrow[dl, "\mathsf{Emp}_{R,\Sigma'}"] \\
 & \mathbf{Set} &
\end{tikzcd} \tag{6}$$

In this way, we obtain a functor

$$\mathsf{Emp}_{R,-}\colon \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Cat} /\!\!/ \mathbf{Set}.$$

Applying the relative Grothendieck construction (Definition A.3), we obtain the empirical model functor on the category of event scenarios.

Definition 2.9. We define the *R-empirical model functor* to be the relative Grothendieck construction of $\mathsf{Emp}_{R,-}$:

$$\mathsf{Emp}_R = \int_{\mathbf{Rel}^{\mathrm{op}}} \mathsf{Emp}_{R,-}\colon \mathbf{eScen} \to \mathbf{Set}.$$

In the case that $R = \mathbb{R}_{\geq 0}$ Definition 2.9 coincide with [11][Definition 2.15]. Then, the codomain of the empirical model functor is **Conv** the category of convex sets [16]. For an event scenario F, we will write $\mathsf{Emp}(F)$ for $\mathsf{Emp}_{\mathbb{R}_{\geq 0}}(F)$, and call the elements of $\mathsf{Emp}_{\mathbb{R}_{\geq 0}}(F)$ empirical models, and call the elements of $\mathsf{Emp}_{\mathbb{B}}(F)$ *possibilistic empirical models.*

Remark 2.10. For an event scenario $F\colon \mathbf{C}_{\Sigma}^{\mathrm{op}} \to \mathbf{Set}$, an element in $p \in \mathsf{Emp}_R(F)$ is a collection of distributions $\{p_\sigma\}_{\sigma\in\Sigma}$ such that

- For every $\sigma \in \Sigma$, we have $p_\sigma \in D_R(F(\sigma))$.
- For every morphism $i\colon \sigma \hookrightarrow \tau$, we have $D_R(F(i))(p_\tau) = p_\sigma$.

Usually, we will denote $D_R(F(i))(p_\tau)$ by $p_\tau|_\sigma$.

Definition 2.11. For F an event scenario, we define the canonical map

$$\Theta_F \colon D_R(\lim F) \to \lim\,(D_R \circ F) = \mathsf{Emp}_R(F).$$

An R-empirical model p on F is called *R-contextual* if it does not lie in the image of Θ_F. Otherwise, it is called *R-noncontextual*.

In the case $R = \mathbb{R}_{\geq 0}$, we will simply say *contextual* and *noncontextual* instead of $\mathbb{R}_{\geq 0}$-contextual and $\mathbb{R}_{\geq 0}$-noncontextual.

Example 2.12. We define the standard scenario $(\partial\Delta^2, O = O_{\mathbb{Z}_2})$ (see Examples 2.1 and 2.5), where

- $\partial\Delta^2 = \{\{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}\}$, and
- $O_x = O_y = O_z = \mathbb{Z}_2$.

We define $p \in \mathsf{Emp}(\mathcal{E}_{(\partial\Delta^2, O)})$ by

$$p^{\vec{a}}_{\{x,y\}} = p^{\vec{a}}_{\{x,z\}} = \begin{cases} \frac{1}{2}, & \text{if } \vec{a} \in \{(0,0),(1,1)\}, \\ 0, & \text{otherwise,} \end{cases} \qquad p^{\vec{a}}_{\{y,z\}} = \begin{cases} \frac{1}{2}, & \text{if } \vec{a} \in \{(1,0),(0,1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

The empirical model p is contextual. This model can be viewed as an analogue of the well-known *PR box* model (see [1, 17]), with the underlying measurement scenario given by triangle instead of square. In both cases, contextuality arises from the incompatibility between pairwise correlations and global consistency.

2.3 Simplicial bundle scenarios

In this subsection, we introduce the notion of simplicial bundle scenarios, which generalize event-based measurement scenarios to the simplicial setting. To this end, we begin with the standard notion of a simplicial set, as commonly used in algebraic topology.

A *simplicial set* is a functor

$$X \colon \Delta^{\mathrm{op}} \to \mathbf{Set},$$

where Δ is the simplicial category whose objects are the finite ordered sets

$$[n] = \{0 < 1 < \cdots < n\}, \quad n \geq 0,$$

and whose morphisms are order-preserving maps, called *ordinal maps*.

For each $n \geq 0$, the set $X_n := X([n])$ is called the set of *n-simplices* of X. The images of the coface maps d^i and codegeneracy maps s^j in Δ under X are called the *face maps* and *degeneracy maps*, respectively:

$$d_i = (d^i)^* \colon X_n \to X_{n-1}, \qquad s_j = (s^j)^* \colon X_n \to X_{n+1},$$

which satisfy the standard simplicial identities (see [18]). A simplex $\sigma \in X_n$ is called a *generator* if it is not in the image of any face or degeneracy map.

A *simplicial set map* (or *simplicial map*) between simplicial sets

$$f \colon X \to Y$$

is a natural transformation between the corresponding functors $X, Y \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$. Equivalently, it consists of a family of functions $\{f_n \colon X_n \to Y_n\}_{n \geq 0}$, commute with all face and degeneracy maps of X and Y. For $x \in X_n$, we will usually write f_x to denote the value of $f_n(x)$.

Definition 2.13. A *circle* C is a simplicial set specified by a sequence of pairwise distinct 1-simplices $\sigma_1, \cdots, \sigma_n \in C$ satisfying

$$d_0(\sigma_1) = d_1(\sigma_2)\,,\ d_0(\sigma_2) = d_1(\sigma_3)\,, \cdots,\ d_0(\sigma_{n-1}) = d_1(\sigma_n)\,,\ d_0(\sigma_n) = d_1(\sigma_1)$$

We sometimes write $C = C^{(n)}$ to indicate that the circle has n edges.

Here is a simplicial set version of Definition 2.2.

Definition 2.14. For natural numbers n and m, let $A_1, \ldots, A_n$ be sets satisfying:

- $|A_i| = m$ for every $1 \leq i \leq n$.
- $A_i \cap A_j = \emptyset$ for every $1 \leq i < j \leq n$.

We define $sB(n,m)$ to be the simplicial set that generated by:

$$A_1 \times A_2 \times \cdots \times A_n,$$

where the face and degeneracy maps given by deleting and repeating coordinates, respectively.

Definition 2.15. Let $m \geq 1$. The simplicial set $\Delta_{\mathbb{Z}_m}$ is defined by

$$(\Delta_{\mathbb{Z}_m})_n = \mathbb{Z}_m^{n+1} \qquad \text{for all } n \geq 0,$$

with face and degeneracy maps given as follows:

For $0 \leq i \leq n$, the i-th face map $d_i \colon \mathbb{Z}_m^{n+1} \longrightarrow \mathbb{Z}_m^n$ is defined by deleting the i-th component:

$$d_i(a_0, \ldots, a_n) = (a_0, \ldots, \widehat{a_i}, \ldots, a_n).$$

For $0 \leq i \leq n$, the i-th degeneracy map $s_i \colon \mathbb{Z}_m^{n+1} \longrightarrow \mathbb{Z}_m^{n+2}$ is defined by repeating the i-th component:

$$s_i(a_0, \ldots, a_n) = (a_0, \ldots, a_i, a_i, \ldots, a_n).$$

Definition 2.16. Given simplicial maps $f \colon E \to X$ and $g \colon Y \to Z$. We say that the map f has the right-lifting property with respect to g, if the diagonal map always exists in the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & E \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

Definition 2.17. A simplicial map $f \colon E \to X$, where $|E_n| < \infty$ for every $n \geq 0$, is called a *simplicial bundle scenario* (or simply a *bundle scenario*) if it is *surjective*, i.e. the map

$$f_n \colon E_n \twoheadrightarrow X_n$$

is surjective for all $n \geq 0$. The simplicial set X is called the *base space* of f, and the simplicial set E the *event space*.

The bundle scenario f is called a *proper simplicial bundle scenario* (or simply a *proper bundle scenario*) if, in addition, it satisfies the following lifting conditions:

1. *Locally surjective:* f has the right-lifting property with respect to all face maps $d_i \colon \Delta[n-1] \hookrightarrow \Delta[n]$.

2. *Discrete over vertices:* f has the right-lifting property with respect to all degeneracy maps $s_i\colon \Delta[n] \to \Delta[n-1]$.

Alternatively, the map f is a proper bundle scenario if for any ordinal map $\theta\colon [m] \to [n]$, it has the right-lifting property with respect to the induced simplicial map $\theta^*\colon \Delta[n] \to \Delta[m]$. This properties first introduced in [7, Definition 4.1].

Example 2.18. Given a simplicial set X and $m \geq 2$. The projection

$$f_{X,m}\colon X \times \Delta_{\mathbb{Z}_m} \longrightarrow X.$$

will be our main example of a simplicial bundle scenario.

Definition 2.19. For a simplicial set X, we define the category $\mathsf{sBund}(X)$ to be:

- objects are simplicial bundle scenarios (see Definition 2.17).
- a morphism from $f\colon E \to X$ to $g\colon F \to X$ is a simplicial map $\alpha\colon E \to F$ that makes the following diagram commute:

$$\begin{tikzcd}
E \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & F \arrow[dl, "g"] \\
& X &
\end{tikzcd} \tag{7}$$

In other words, $\mathsf{sBund}(X)$ is the subcategory of the slice category $\mathbf{sSet}/X$, where the objects are the simplicial bundle scenarios.

Definition 2.20. For a simplicial map $\pi\colon Y \to X$, we define the functor $\pi^*\colon \mathsf{sBund}(X) \to \mathsf{sBund}(Y)$ by

- sending an object $f\colon E \to X$ in $\mathsf{sBund}(X)$ to the object $\pi^*(f)$ that defined by the pullback square:

$$\begin{tikzcd}
E \arrow[d, "f"] & E \times_X Y \arrow[l] \arrow[d, "\pi^*(f)"] \arrow[dl, phantom, "\llcorner", very near start] \\
X & Y \arrow[l, "\pi"]
\end{tikzcd}$$

  Here, $\pi^*(f)$ is the canonical projection from the pullback.

- sending a morphism α as in Diagram (7), to the morphism $\pi^*(\alpha) = \alpha \times \mathrm{id}_Y$:

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[d, "f"] & F \arrow[d, "g"] & & \\
X \arrow[r, equal] & X & E \times_X Y \arrow[r, "\alpha\times\mathrm{id}_Y"] \arrow[ull] \arrow[d, "\pi^*(f)"] & F \times_X Y \arrow[ull] \arrow[d, "\pi^*(g)"] \\
& & Y \arrow[r, equal] \arrow[ull, "\pi"] & Y \arrow[ull, "\pi"]
\end{tikzcd}$$

Note that since f is surjective, the map $\pi^*(f)$ is also surjective, so it belongs to $\mathsf{sBund}(Y)$. By Definitions 2.19 and 2.20 we have a functor

$$\mathsf{sBund}\colon \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Cat}$$

Definition 2.21. We define the category of *simplicial bundle secnario*, denoted by **sScen**, to be the Grthendiek construction $\int_{\mathbf{sSet}^{\mathrm{op}}} \mathsf{sBund}$.

Definition 2.21 coincides with the definition given in [7, Section 4].

2.4 Simplicial distributions on bundle scenarios

In this subsection, we define the notion of simplicial distributions on bundle scenarios, which extend empirical models (see [7]).

The functoriality of the distribution monad D_R (Definition 2.8) allows it to extend to a monad on simplicial sets, $D_R\colon \mathbf{sSet} \to \mathbf{sSet}$. In this case, for a simplicial set X, the unit $\delta_X\colon X \to D_R(X)$ acts as follows:

$$(\delta_X)_n(x) = \delta_{X_n}(x) = \delta^x$$

for any $x \in X_n$; see Equation (4).

Definition 2.22. Let $f\colon E \to X$ be a simplicial bundle scenario. An *R-simplicial distribution* on f is a simplicial set map $p\colon X \to D_R(E)$ making the following diagram commute:

$$\begin{tikzcd} & D_R(E) \arrow[d, "D_R(f)"] \\ X \arrow[ur, "p"] \arrow[r, "\delta_X"'] & D_R(X) \end{tikzcd} \tag{8}$$

Proposition 2.23. *([7, Proposition 4.9]) A simplicial map $p\colon X \to D_R(E)$ is an R-simplicial distribution if and only if for every $x \in X_n$ we have:*

$$\{e \in E_n :\ p_n(x)(e) \neq 0\} \subseteq f_n^{-1}(x)$$

Corollary 2.24. *Let p be a simplicial distribution on a simplicial bundle scenario $f\colon E \to X$. If σ is a generator simplex in X_n, then*

$$p_\sigma(d_i(e_1)) = 0 \quad \textit{and} \quad p_\sigma(s_j(e_2)) = 0$$

for all $e_1 \in E_{n+1}$ and $e_2 \in E_{n-1}$.

Proof. Suppose $p_\sigma(d_i(e_1)) \neq 0$ for some $e_1 \in E_{n+1}$. By Proposition 2.23, this implies $f_n(d_i(e_1)) = \sigma$, hence $d_i(f_{n+1}(e_1)) = \sigma$. This contradicts the assumption that σ is a generator simplex. Therefore, $p_\sigma(d_i(e_1)) = 0$. A similar argument applies to degeneracies. □

For every simplicial set X, we define a functor

$$\mathsf{sDist}_R(X)\colon \mathsf{sBund}(X) \to \mathbf{Set}$$

that sends a simplicial bundle scenario $f \in \mathsf{sBund}(X)$ to the set of R-simplicial distributions on f, denoted by $\mathsf{sDist}_R(X)(f)$. Given a morphism $\alpha\colon f \to g$ in $\mathsf{sBund}(X)$, the functor $\mathsf{sDist}_R(X)$ acts by

$$\alpha_*\colon \mathsf{sDist}_R(X)(f) \longrightarrow \mathsf{sDist}_R(X)(g), \qquad p \longmapsto D_R(\alpha) \circ p.$$

Furthermore, for every simplicial map $\pi\colon Y \to X$, we obtain a natural transformation in the slice category $\mathbf{Cat}/\!\!/\mathbf{Set}$:

$$\begin{tikzcd}
\mathsf{sBund}(X) \arrow[rr, "\pi^*"] \arrow[dr, "\mathsf{sDist}_R(X)"', ""{name=A}] & & \mathsf{sBund}(Y) \arrow[dl, "\mathsf{sDist}_R(Y)", ""'{name=B}] \\
& \mathbf{Set} &
\arrow[Rightarrow, from=A, to=B, "\pi^*_-"]
\end{tikzcd}$$

where for each $f \in \mathsf{sBund}(X)$, the natural map $\pi^*_f\colon \mathsf{sDist}_R(X)(f) \to \mathsf{sDist}_R(Y)(\pi^*(f))$ sends p to $\pi_*(p)$ as in [7, Definition 4.11]. More explicitly,

$$\pi^*_f(p)(y)(e,y) = p_n(\pi_n(y))(e) \tag{9}$$

for every $(e,y) \in E_n \times_{X_n} Y_n$, see [7, Equation (22)]. Thus, we have a functor

$$\mathsf{sDist}_{R,-}\colon \mathbf{sSet}^{\mathrm{op}} \longrightarrow \mathbf{Cat}/\!\!/\mathbf{Set}.$$

Definition 2.25. The *R-simplicial distribution functor* is defined as the relative Grothendieck construction of $\mathsf{sDist}_{R,-}$:

$$\mathsf{sDist}_R = \int_{\mathbf{sSet}^{\mathrm{op}}} \mathsf{sDist}_{R,-}\colon \mathbf{sScen} \longrightarrow \mathbf{Set}.$$

When $R = \mathbb{R}_{\geq 0}$, we call $\mathsf{sDist}_{\mathbb{R}_{\geq 0}}$, the *simplicial distributions functor*, and denote it by sDist. Meanwhile, when $R = \mathbb{B}$, we call $\mathsf{sDist}_{\mathbb{B}}$, the *possibilistic simplicial distributions functor*.

The notion of contextuality also extends to the simplicial set setting. Given a simplicial bundle scenario $f\colon E \to X$, let $\mathsf{sSect}(f)$ denote the set of sections of f. We have a map

$$\Theta_f\colon D_R(\mathsf{sSect}(f)) \longrightarrow \mathsf{sDist}_R(f) \tag{10}$$

as described in [7, (29)].

Definition 2.26. Given a simplicial bundle scenario $f\colon E \to X$. An R-simplicial distribution p on f is called *R-contextual* if it does not lie in the image of Θ_f. Otherwise, it is called *R-noncontextual*. When $R = \mathbb{R}_{\geq 0}$, we write simply *contextual* and *noncontextual*.

2.5 Extremal contextual models

For an event scenario F and a simplicial bundle scenario f, both $\mathsf{Emp}(F)$ and $\mathsf{sDist}(f)$ are convex sets. In this section, we give a main tool for characterizing vertices of $\mathsf{Emp}(F)$ and $\mathsf{sDist}(f)$.

Definition 2.27. An element v in a convex set V is called a *vertex (extremal point)* if for every $0 < t < 1$ and elements $v_1, v_2 \in V$ such that $v = t \cdot v_1 + (1-t) \cdot v_2$, we have $v = v_1 = v_2$.

Example 2.28. Given the simplicial bundle scenario $f_{C^{(n)},m}\colon C^{(n)} \times \Delta_{\mathbb{Z}_m} \to C^{(n)}$ (Example 2.18). For $1 \leq k \leq m$, a simplicial distribution

$$p\colon C^{(n)} \to D(C^{(n)} \times \Delta_{\mathbb{Z}_m})$$

is called a *k-order cycle distribution* on $f_{C^{(n)},m}$ if there exists a finite sequence

$$\left(a_1^{(1)}, \dots, a_n^{(1)};\ a_1^{(2)}, \dots, a_n^{(2)};\ \dots;\ a_1^{(k)}, \dots, a_n^{(k)}\right)$$

of elements in $\mathbb{Z}_m$ such that $a_i^{(j)} \neq a_i^{(s)}$ for every $1 \leq i \leq n$ and $j \neq s$, and the distribution is defined by

$$p_{\sigma_i}^{(\sigma_i,(a,b))} = \begin{cases} \dfrac{1}{k} & \text{if } (a,b) = \left(a_i^{(j)},\, a_{i+1}^{(j)}\right) \text{ for some } 1 \leq j \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq i \leq n-1$, and

$$p_{\sigma_n}^{(\sigma_n,(a,b))} = \begin{cases} \dfrac{1}{k} & \text{if } (a,b) = \left(a_n^{(j)},\, a_1^{(j+1)}\right) \text{ for some } 1 \leq j \leq k-1, \\ \dfrac{1}{k} & \text{if } (a,b) = \left(a_n^{(k)},\, a_1^{(1)}\right), \\ 0 & \text{otherwise.} \end{cases}$$

According to [9, Corollary 4.7] p is a vertex.

The empirical model in Example 2.12 corresponds to a 2-order cycle on a 3-cycle scenario. Therefore, it is a vertex of the polytope $\mathsf{Emp}(\mathcal{E}_{(\partial\Delta^2,O)})$. We now present another example of a contextual vertex.

Example 2.29. The simplicial complex $B(2,2)$ (Definition 2.2) is generated by the simplices

$$\{x,y\},\ \ \{x,y'\},\ \ \{x',y\},\ \ \{x',y'\}.$$

Consider the measurement scenario $(B(2,2), O_{\mathbb{Z}_2})$ (see Example 2.5). We define $p \in \mathsf{Emp}(\mathcal{E}_{(B(2,2),O_{\mathbb{Z}_2})}$ by setting:

$$p_{\{x,y'\}}^{\vec{a}} = p_{\{x',y\}}^{\vec{a}} = p_{\{x',y'\}}^{\vec{a}} = \begin{cases} \frac{1}{2} & \text{if } \vec{a} \in \{(0,0),(1,1)\} \\ 0 & \text{otherwise,} \end{cases}$$

$$p_{\{x,y\}}^{\vec{a}} = \begin{cases} \frac{1}{2} & \text{if } \vec{a} \in \{(1,0),(0,1)\} \\ 0 & \text{otherwise,} \end{cases}$$

The empirical model p is a contextual vertex in $\mathsf{Emp}(\mathcal{E}_{(B(2,2),O_{\mathbb{Z}_2})})$ (see [19]). Moreover, it can be viewed as a 2-order cycle on a 4-cycle scenario.

Definition 2.30. Let $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$ be an event scenario. We define a preorder $\preceq$ on $\mathsf{Emp}_R(F)$ by declaring that $q \preceq p$ if, for every $\sigma \in \Sigma$ and every $a \in F(\sigma)$,

$$q_\sigma^a \neq 0 \ \ \Rightarrow\ \ p_\sigma^a \neq 0.$$

We denote by $p_{\preceq}$ the set of all empirical models $q \in \mathsf{Emp}_R(F)$ such that $q \preceq p$.

Definition 2.31. Let $f\colon E \to X$ be a simplicial bundle scenario. We define a preorder $\preceq$ on $\mathsf{sDist}_R(f)$ by declaring that $q \preceq p$ if, for every $x \in X_n$ and every $e \in E_n$,

$$q_x^e \neq 0 \ \ \Rightarrow\ \ p_x^e \neq 0.$$

We denote by $p_{\preceq}$ the set of all simplicial distributions $q \in \mathsf{sDist}_R(f)$ such that $q \preceq p$.

We now state two versions of [9, Corollary 2.6] that will be used to characterize vertices.

Proposition 2.32. *An empirical model* $p \in \mathsf{Emp}(F)$ *is a vertex if and only if* p *is minimal with respect to the preorder* $\preceq$*.*

Proposition 2.33. *A simplicial distribution* $p \in \mathsf{sDist}(f)$ *is a vertex if and only if* p *is minimal with respect to the preorder* $\preceq$*.*

3 Characterization of possibilistic empirical models

In this section, we provide a new characterization for the possibilistic empirical models on event scenarios. The characterization depends on the properties of proper event scenarios described in 2.4.

3.1 The functor of sub-event scenarios

In this section, we define the functor of sub-proper event scenarios (or shortly sub-event scenarios) using the relative Grothendieck construction. We begin by defining a partial order on $\mathsf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ for a given category $\mathbf{C}$.

Definition 3.1. Given functors $F', F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, we write $F' \leq F$ if:

- $F'(a) \subseteq F(a)$ for every object a in $\mathbf{C}$.
- For every morphism $s\colon a \to b$ in $\mathbf{C}$, we have $F'(s)(x) = F(s)(x)$ for every $x \in F'(b)$.

Remark 3.2. The second condition in Definition 3.1 is equivalent to requiring that the inclusions $\{F'(a) \hookrightarrow F(a)\}_{a \in \mathbf{C}}$ form a natural transformation from F' to F.

Definition 3.3. Given functors $F, G\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, and a natural transformation $\alpha\colon F \to G$, we define the functor $\alpha_*(F)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ as follows:

- For an object a, $\alpha_*(F)(a) = \alpha_a(F(a))$.
- For a morphism $s\colon a \to b$, $\alpha_*(F)(s)$ is the restriction

$$G(s)|_{\alpha_b(F(b))}\colon \alpha_b(F(b)) \to \alpha_a(F(a)). \tag{11}$$

The map in (11) is well-defined due to the commutative diagram:

$$\begin{tikzcd}
F(b) \arrow[r, "\alpha_b"] \arrow[d, "F(s)"'] & G(b) \arrow[d, "G(s)"] \\
F(a) \arrow[r, "\alpha_a"'] & G(a)
\end{tikzcd} \tag{12}$$

Note that $\alpha_*(F) \leq G$.

Proposition 3.4. *For proper event scenarios* $F, G\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$*, the functor* $\alpha_*(F)$ *in Definition 3.3 is a proper event scenario.*

Proof. **Non-triviality:** For every $\sigma \in \Sigma$, we have $\alpha_*(F)(\sigma) = \alpha_\sigma(F(\sigma)) \neq \emptyset$, since $F(\sigma) \neq \emptyset$.

Local surjectivity: For $s\colon \sigma \hookrightarrow \tau$, the map

$$G(s)|_{\alpha_\tau(F(\tau))}\colon \alpha_\tau(F(\tau)) \to \alpha_\sigma(F(\sigma))$$

is surjective. This follows from the commutativity of diagram (12) and the fact that $F(s)$ is surjective.

$\square$

Definition 3.5. For every simplicial complex Σ, we define the functor

$$\mathsf{ESub}_\Sigma\colon \mathsf{Event}(\Sigma) \to \mathbf{Set}$$

by:

- For an object $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$ in $\mathsf{Event}(\Sigma)$, define

$$\mathsf{ESub}_\Sigma(F) = \left\{F'\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set} \;\middle|\; F' \leq F \text{ and } F' \text{ is a proper event scenario}\right\}.$$

- For a natural transformation $\alpha\colon F \to G$, define

$$\alpha_* = \mathsf{ESub}_\Sigma(\alpha)\colon \mathsf{ESub}_\Sigma(F) \to \mathsf{ESub}_\Sigma(G)$$

by sending $F' \in \mathsf{ESub}_\Sigma(F)$ to the functor

$$(\alpha \circ j)_*(F')\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set},$$

where $j\colon F' \to F$ is the natural transformation given by the inclusions $F'(\sigma) \hookrightarrow F(\sigma)$.

See Remark 3.2, Definition 3.3, and Proposition 3.4.

Note that if we have a simplicial complex map $\pi\colon \Sigma' \to \hat{N}\Sigma$, and a proper event scenario $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$, then $F' \leq F$ implies $\pi^*(F') \leq \pi^*(F)$.

Definition 3.6. For a simplicial complex map $\pi\colon \Sigma' \to \hat{N}\Sigma$ and an event scenario $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$, we define a map

$$\mathsf{ESub}_\pi(F)\colon \mathsf{ESub}_\Sigma(F) \to \mathsf{ESub}_{\Sigma'}(\pi^*(F)) \tag{13}$$

by sending $F' \in \mathsf{ESub}_\Sigma(F)$ to $\pi^*(F') \in \mathsf{ESub}_{\Sigma'}(\pi^*(F))$.

Given $F, G \in \mathsf{Event}$, and a natural transformation $\alpha\colon F \to G$, one can check that the following diagram commute:

$$\begin{tikzcd}
\mathsf{ESub}_\Sigma(F) \arrow[r, "\mathsf{ESub}_\pi(F)"] \arrow[d, "\alpha_*"] & \mathsf{ESub}_{\Sigma'}(\pi^*(F)) \arrow[d, "(\mathrm{id}_{\overline{\pi}} \star \alpha)_*"] \\
\mathsf{ESub}_\Sigma(G) \arrow[r, "\mathsf{ESub}_\pi(G)"] & \mathsf{ESub}_{\Sigma'}(\pi^*(G))
\end{tikzcd}$$

Therefore, the maps in (13) form a natural transformation from ESub_Σ to $\mathsf{ESub}_{\Sigma'} \circ \pi^*$.

For a simplicial complex map $\pi\colon \Sigma' \to \hat{N}\Sigma$, we define the following morphism in $\mathbf{Cat}/\!\!/\mathbf{Set}$:

$$\begin{tikzcd}
\mathsf{Event}(\Sigma) \arrow[rr, "\pi^*"] \arrow[dr, "\mathsf{ESub}_\Sigma"', ""{name=A}] & & \mathsf{Event}(\Sigma') \arrow[dl, "\mathsf{ESub}_{\Sigma'}", ""'{name=B}] \\
& \mathbf{Set} &
\arrow[Rightarrow, from=A, to=B, "\mathsf{ESub}_\pi"]
\end{tikzcd} \tag{14}$$

Thus, we obtain a functor

$$\mathsf{ESub}_{-}\colon \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Cat}/\!\!/\mathbf{Set}$$

that sends a simplicial complex Σ to the functor $\mathsf{ESub}_{\Sigma}\colon \mathsf{Event}(\Sigma) \to \mathbf{Set}$, and sends a simplicial complex map $\pi\colon \Sigma' \to \hat{N}\Sigma$ to the diagram (14).

Definition 3.7. The *sub-proper event scenarios functor* or shortly the *sub-event scenarios functor* is defined to be the relative Grothendieck construction of ESub_{-}:

$$\mathsf{ESub} = \int_{\mathbf{Rel}^{\mathrm{op}}} \mathsf{ESub}_{-}\colon \mathbf{eScen} \to \mathbf{Set}.$$

See Definitions A.3 and 2.7.

3.2 Possibilistic empirical models as sub-event scenarios

In this section, we show that the functor of sub-event scenarios is equivalent to the functor of possibilistic empirical models defined in section 2.2. This done using a relative version of the Grothendieck construction (Section A).

Proposition 3.8. *Given an event scenario $F\colon \mathbf{C}_{\Sigma}^{\mathrm{op}} \to \mathbf{Set}$ and $p \in \mathsf{Emp}_{\mathbb{B}}(F)$. Let*

$$L(p)\colon \mathbf{C}_{\Sigma}^{\mathrm{op}} \to \mathbf{Set}$$

be the functor that sends $\sigma \in \mathbf{C}_{\Sigma}$ to $L(p)(\sigma) = \{x \in F(\sigma) : p_{\sigma}(x) = 1\}$. Then, $L(p)$ is a proper event scenario such that $L(p) \leq F$.

Proof. First, we prove that $L(p)$ is a well-defined functor. For a map $i\colon \sigma \hookrightarrow \tau$ in $\mathbf{C}_{\Sigma}$, suppose $x \in L(p)(\tau)$, meaning that $p_{\tau}(x) = 1$. Then:

$$\begin{aligned}
p_{\sigma}(F(i)(x)) &= (p_{\tau}|_{\sigma})\,(F(i)(x)) \\
&= D_{\mathbb{B}}(F(i))(p_{\tau})\,(F(i)(x)) \\
&= \sum_{y\colon F(i)(y)=F(i)(x)} p_{\tau}(y) \\
&= p_{\tau}(x) + \sum_{y\neq x,\, F(i)(y)=F(i)(x)} p_{\tau}(y) \\
&= 1.
\end{aligned}$$

See Remark 2.10.

Non-triviality: For each simplex $\sigma \in \Sigma$, since $\sum_{x\in F(\sigma)} p_{\sigma}(x) = 1$, there exists some $x' \in F(\sigma)$ such that $p_{\sigma}(x') = 1$. Hence, $x' \in L(p)(\sigma)$, so $L(p)(\sigma) \neq \emptyset$.

Local surjectivity: For $i\colon \sigma \hookrightarrow \tau$, consider $y \in L(p)(\sigma)$. Then:

$$1 = p_{\sigma}(y) = p_{\tau}|_{\sigma}(y) = \sum_{F(i)(x)=y} p_{\tau}(x).$$

So there is $x \in F(\tau)$ such that $F(i)(x) = y$ and $p_{\tau}(x) = 1$.

□

Definition 3.9. For an event scenario $F\colon \mathbf{C}_{\Sigma}^{\mathrm{op}} \to \mathbf{Set}$, we define the map

$$\eta_{\Sigma,F}\colon \mathsf{Emp}_{\mathbb{B},\Sigma}(F) \to \mathsf{ESub}_{\Sigma}(F) \tag{15}$$

by sending $p \in \mathsf{Emp}_{\mathbb{B},\Sigma}(F)$ to the proper event scenario $L(p)$.

Proposition 3.10. *The maps $\eta_{\Sigma,F}$ from Definition 3.9 form a natural isomorphism from $\mathsf{Emp}_{\mathbb{B},\Sigma}$ to ESub_{Σ}.*

Proof. To prove naturality, consider a natural transformation $\alpha\colon F \to G$, where $F, G\colon \mathbf{C}_{\Sigma}^{\mathrm{op}} \to \mathbf{Set}$ are event scenarios. We show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Emp}_{\mathbb{B},\Sigma}(F) \arrow[r, "\lim(\alpha\star\mathrm{id}_{D_{\mathbb{B}}})"] \arrow[d, "\eta_{\Sigma,F}"'] & \mathsf{Emp}_{\mathbb{B},\Sigma}(G) \arrow[d, "\eta_{\Sigma,G}"] \\
\mathsf{ESub}_{\Sigma}(F) \arrow[r, "\alpha_*"'] & \mathsf{ESub}_{\Sigma}(G)
\end{tikzcd}$$

For $p \in \mathsf{Emp}_{\mathbb{B},\Sigma}(F)$ and $\sigma \in \Sigma$, we have:

$$\begin{aligned}
\eta_{\Sigma,G}\left(\lim(\alpha \star \mathrm{id}_{D_{\mathbb{B}}})(p)\right)(\sigma) &= \{y \in G(\sigma) :\ D_{\mathbb{B}}(p_\sigma)(y) = 1\} \\
&= \{y \in G(\sigma) :\ \sum_{\alpha_\sigma(x)=y} p_\sigma(x) = 1\} \\
&= \{y \in G(\sigma) :\ \exists x \in F(\sigma)\ \text{ s.t }\ \alpha_\sigma(x) = y\ \text{ and }\ p_\sigma(x) = 1\}
\end{aligned}$$

On the other hand, we have:

$$\alpha_*\left(\eta_{\Sigma,F}(p)\right)(\sigma) = \alpha_\sigma\left(\eta_{\Sigma,F}(p)(\sigma)\right) = \alpha_\sigma(\{x \in F(\sigma) :\ p_\sigma(x) = 1\}) = \{\alpha_\sigma(x) :\ p_\sigma(x) = 1\}$$

Next, we prove that $\eta_{\Sigma,F}$ is an isomorphism. To show surjectivity, let $F' \in \mathsf{ESub}_{\Sigma}(F)$. For each $\sigma \in \Sigma$, define:

$$p_\sigma(x) = \begin{cases} 1 & \text{if } x \in F'(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $p = \{p_\sigma\}_{\sigma\in\Sigma}$ belongs to $\mathsf{Emp}_{\mathbb{B},\Sigma}(F)$. Since F' is non-trivial, there exists $x \in F'(\sigma)$, so $p_\sigma(x) = 1$. Given a morphism $\sigma \hookrightarrow \tau$ and $y \in F(\sigma)$, assume $p_\sigma(y) = 1$. Then $y \in F'(\sigma)$. Since F' is locally surjective, there exists $\tilde{x} \in F'(\tau)$ such that $F'(i)(\tilde{x}) = y$. Therefore,

$$p_\tau|_\sigma(y) = \sum_{F(i)(x)=y} p_\tau(x) = p_\tau(\tilde{x}) + \cdots = 1.$$

Conversely, if $p_\tau|_\sigma(y) = 1$, then there exists $x \in F(\tau)$ such that $F(i)(x) = y$ and $p_\tau(x) = 1$. This implies $x \in F'(\tau)$, and since $F' \leq F$, we have $F'(i)(x) = y$. In particular, $y \in F'(\sigma)$, so $p_\sigma(y) = 1$. Now, observe that:

$$\eta_{\Sigma,F}(p)(\sigma) = \{x \in F(\sigma) :\ p_\sigma(x) = 1\} = \{x \in F(\sigma) :\ x \in F'(\sigma)\} = F'(\sigma)$$

For injectivity, given $p, q \in \mathsf{Emp}_{\mathbb{B},\Sigma}(F)$ such that $\eta_{\Sigma,F}(p) = \eta_{\Sigma,F}(q)$. Then for every $\sigma \in \Sigma$:

$$\{x \in F(\sigma) :\ p_\sigma(x) = 1\} = \{x \in F(\sigma) :\ q_\sigma(x) = 1\},$$

This equality implies that $p = q$. ☐

Proposition 3.11. *The transformation η is a natural isomorphism from $\mathsf{Emp}_{\mathbb{B},-}$ to ESub_{-}.*

Proof. By Proposition 3.10, for every Σ object in $\mathbf{Rel}^{\mathrm{op}}$ we have the isomorphism $(\mathrm{id}_{\mathsf{Event}(\Sigma)}, \eta_\Sigma)$ in $\mathbf{Cat}/\!\!/\mathbf{Set}$:

$$\begin{tikzcd}
\mathsf{Event}(\Sigma) \arrow[rr, equal] \arrow[dr, "\mathsf{Emp}_{\mathbb{B},\Sigma}"', ""{name=A}] & & \mathsf{Event}(\Sigma) \arrow[dl, "\mathsf{ESub}_{\Sigma}", ""'{name=B}] \\
& \mathbf{Set} &
\arrow[Rightarrow, from=A, to=B, "\eta_\Sigma"]
\end{tikzcd}$$

Given a simplicial complex map $\pi\colon \Sigma' \to \hat{N}\Sigma$, we need to prove that the following diagram commutes in $\mathbf{Cat}/\!\!/\mathbf{Set}$:

$$\begin{tikzcd}
\mathsf{Emp}_{\mathbb{B},\Sigma} \arrow[r, "(\pi^*{,}\mathsf{Emp}_{\mathbb{B},\pi})"] \arrow[d, "(\mathrm{id}_{\mathsf{Event}(\Sigma)}{,}\eta_\Sigma)"'] & \mathsf{Emp}_{\mathbb{B},\Sigma'} \arrow[d, "(\mathrm{id}_{\mathsf{Event}(\Sigma')}{,}\eta_{\Sigma'})"] \\
\mathsf{ESub}_{\Sigma} \arrow[r, "(\pi^*{,}\mathsf{ESub}_{\pi})"'] & \mathsf{ESub}_{\Sigma'}
\end{tikzcd}$$

Given an event scenario $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$, and an empirical model $p = \{p_\sigma\}_{\sigma\in\Sigma} \in \mathsf{Emp}_{\mathbb{B}}(F)$. We denote $\mathsf{Emp}_{\mathbb{B},\pi}(F)(p) = \{p_{\overline{\pi}(\tau)}\}_{\tau\in\Sigma'}$ by q, then we have:

$$\eta_{\Sigma',F}\left(\mathsf{Emp}_{\mathbb{B},\pi}(F)(p)\right) = \eta_{\Sigma',F}(q) = L(q),$$

and $L(q)(\tau) = \{x \in F(\overline{\pi}(\tau)) :\ q_\tau(x) = 1\} = \{x \in F(\overline{\pi}(\tau)) :\ p_{\overline{\pi}(\tau)}(x) = 1\}$ for every $\tau \in \Sigma'$. On the other hand, we have

$$\mathsf{ESub}_\pi(F)\left(\eta_{\Sigma,F}(p)\right) = \pi^*\left(L(p)\right) = L(p) \circ \overline{\pi},$$

and $L\left(\overline{\pi}(p)(\tau)\right) = \{x \in F(\overline{\pi}(\tau)) :\ p_{\overline{\pi}(\tau)}(x) = 1\}$ for every $\tau \in \Sigma'$. □

Theorem 3.12. *The functors* $\mathsf{Emp}_{\mathbb{B}}\colon \mathbf{eScen} \to \mathbf{Set}$ *and* $\mathsf{ESub}\colon \mathbf{eScen} \to \mathbf{Set}$ *are equivalent.*

Proof. Directly by Propositions 3.11 and A.4. □

Proposition 3.13. *For an event scenario* $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$, *the posets* $(\mathsf{Emp}_{\mathbb{B}}(F), \preceq)$ *and* $(\mathsf{ESub}(F), \leq)$ *are isomorphic (see Definitions 3.1 and 2.30).*

Proof. The isomorphism in (15) preserves the preorders. □

4 Categorical characterization of extremal empirical models

In this section, we characterize extremal empirical models in terms of their projections to the Boolean semiring. Then, we use the characterization of possibilistic empirical models from the previous section to detect contextual vertices for some important scenarios.

4.1 The possibilistic structure of extremal empirical models

We demonstrate that contextual extremal empirical models are determined by their corresponding possibilistic empirical models, a fact previously noted in [20].

Definition 4.1. ([8, Definition 7]) Given a convex set V. A subset $U \subseteq V$ is called a *prime filter* if U is a convex subset such that whenever a convex combination $\sum_{i=1}^{n} \alpha_i v_i$ of elements $v_i \in V$ lies in U, where $0 < \alpha_i < 1$ for every $1 \le i \le n$, we get that $v_i \in U$ for every $1 \le i \le n$.

Proposition 4.2. *Let U be a prime filter in a convex set V. An element $v \in U$ is a vertex in V if and only if it is a vertex in U.*

Proof. If $v \in U$ is a vertex in V, then obviously it is a vertex in U. Now, suppose that v is a vertex in U. If there is $v_1, v_2 \in V$ and $0 < \alpha < 1$ such that $v = \alpha v_1 + (1-\alpha)v_2$, then $v_1, v_2 \in U$ since U is a prime filter. So $v_1 = v_2 = v$ since v is a vertex in U. □

For an event sceanrio F, the projection in (5) induces a projection

$$\kappa_F \colon \mathsf{Emp}(F) \to \mathsf{Emp}_{\mathbb{B}}(F),$$

which obviously satisfying $p' \preceq p$ if and only if $\kappa_F(p') \preceq \kappa_F(p)$.

Proposition 4.3. *For a possibilistic empirical model $p \in \mathsf{Emp}_{\mathbb{B}}(F)$, the subset $\kappa_F^{-1}(p_{\preceq})$ is a prime filter which is isomorphic to $\mathsf{Emp}(L(p))$ (see Definition 2.30 and Proposition 3.8).*

Proof. Given $q, s \in \mathsf{Emp}(F)$ and $0 < \alpha < 1$ such that $\alpha q + (1-\alpha)s \in \kappa_F^{-1}(p_{\preceq})$. That means $\kappa_F(\alpha q + (1-\alpha)s) \preceq p$. Since $\alpha > 0$, we get that $\kappa_F(q) \preceq p$, and since $1 - \alpha > 0$, we get that $\kappa_F(s) \preceq p$. So $q, s \in \kappa_F^{-1}(p_{\preceq})$. Now, note that for $q \in \mathsf{Emp}(F)$, we have $q \in \mathsf{Emp}(L(p))$ if and only if for every $\sigma \in \Sigma$ and $y \in F(\sigma)$ the condition $q_\sigma(y) \neq 0$ implies that $y \in L(p(\sigma))$, which means that $p_\sigma(y) = 1$. This is equivalent to say that $\kappa_F(q) \preceq p$. In other words, $q \in \kappa_F^{-1}(p_{\preceq})$. □

The next proposition is a special case of [20, part (3) of Corollary 5.5]:

Proposition 4.4. *Given an event scenario F. An empirical model $p \in \mathsf{Emp}(F)$ is a vertex if and only if $\kappa_F^{-1}(\kappa_F(p)) = \{p\}$.*

Proof. Suppose that p is a vertex. If $p' \in \kappa_F^{-1}(\kappa_F(p)_{\preceq})$, then $p' \preceq p$. By Proposition 2.32 we obtain $p' = p$. We proved that $\kappa_F^{-1}(\kappa_F(p)_{\preceq}) = \{p\}$, so $\kappa_F^{-1}(\kappa_F(p)) = \{p\}$. Now, suppose that $\kappa_F^{-1}(\kappa_F(p)) = \{p\}$. Given a vertex $p' \in \kappa_F^{-1}(\kappa_F(p)_{\preceq})$. By Propositions 4.2 and 4.3, we get that p' is a vertex in $\mathsf{Emp}(F)$. $p' \preceq p$, so $\kappa_F(\frac{1}{2}p + \frac{1}{2}p') = \kappa_F(p)$. In other words, $\frac{1}{2}p + \frac{1}{2}p' \in \pi_F^{-1}(\kappa_F(p))$, so $\frac{1}{2}p + \frac{1}{2}p' = p$, which implies that $p' = p$. □

The next proposition is a special case of [20, part (2) of Corollary 5.5]:

Proposition 4.5. *An empirical model $p \in \mathsf{Emp}(F)$ is a vertex if and only if every $q \prec \kappa_F(p)$ is not in the image of κ_F (not liftable).*

Proof. Suppose first that p is a vertex. If there exists $p' \in \mathsf{Emp}(F)$ such that $\kappa_F(p') \prec \kappa_F(p)$, then $p' \prec p$, which contradicts Proposition 2.32.

Conversely, assume that every $q \prec \kappa_F(p)$ is not liftable. Then the prime filter $\kappa_F^{-1}(\kappa_F(p)_{\preceq})$ (see Proposition 4.3) is equal to $\kappa_F^{-1}(\kappa_F(p))$. Let p' be a vertex in the prime filter $\kappa_F^{-1}(\kappa_F(p))$. By Proposition 4.2 p' is a vertex in $\mathsf{Emp}(F)$, and hence, by Proposition 4.4, we have

$$\{p'\} = \kappa_F^{-1}(\kappa_F(p')).$$

Meanwhile, $\kappa_F(p') = \kappa_F(p)$, so it follows that

$$\kappa_F^{-1}(\kappa_F(p')) = \kappa_F^{-1}(\kappa_F(p)).$$

Therefore, $p' = p$, and we conclude that p is a vertex. □

Corollary 4.6. *If $\kappa_F(p)$ is minimal, then p is a vertex.*

Example 4.7. Consider the contextual empirical model described in [21, Table III]. Note that the projection to the Boolean semiring is not minimal, but every possibilistic empirical model preceding it fails to lift to a probabilistic empirical model.

4.2 Extremal empirical models

In this section, we present our main result on extremal empirical models and apply it to detect contextual vertices in the following classes of measurement simplicial complexes:

- $\partial\Delta^n$: the boundary of the n-simplex.
- $B(n, 2)$: the measurement simplicial complex representing the Bell scenario with n parties and two measurements per party.

Lemma 4.8. *Let C be a category and let $F, G\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be non-trivial and locally surjective functors (Definition 2.4) such that $G \leq F$ (see Definition 3.1), and let $s\colon a \to b$ be a morphism in $\mathbf{C}$. Then:*

1. *If $F(b) = G(b)$, then $F(a) = G(a)$.*
2. *If $F(s)$ is isomorphism and $F(a) = G(a)$, then $F(b) = G(b)$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
G(b) \arrow[r, hook] \arrow[d, two heads, "G(s)"] & F(b) \arrow[d, two heads, "F(s)"] \\
G(a) \arrow[r, hook] & F(a)
\end{tikzcd} \tag{16}$$

For (1), if $G(b) = F(b)$, then the surjectivity of $F(s)$ implies $G(a) = F(a)$. For (2), suppose $G(a) = F(a)$. Since $G(s)$ is surjective, for $x \in F(b)$ there exists $y \in G(b)$ such that $G(s)(y) = F(s)(x)$. By the commutativity of Diagram (16), we have $G(s)(y) = F(s)(y)$, hence $F(s)(x) = F(s)(y)$. The injectivity of $F(s)$ then implies $x = y$. □

Proposition 4.9. *Let $\mathbf{C}$ be a finite poset and let $F, G\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be non-trivial and locally surjective functors such that for every maximal objects a and b in $\mathbf{C}$, there exists a zigzag in $\mathbf{C}$*

$$a \leftarrow a_1 \rightarrow a_2 \leftarrow \ldots \rightarrow b \tag{17}$$

where $F(s)$ is an isomorphism for every morphism s in (17). If $G \leq F$, and for a maximal object c in $\mathbf{C}$, we have $G(c) = F(c)$, then $G = F$.

Proof. Let c be a maximal object of $\mathbf{C}$ such that $G(c) = F(c)$. We prove that $G(x) = F(x)$ for every object $x \in \mathbf{C}$. For any maximal object b there exists a zigzag as in (17), such that $F(s)$ is an isomorphism for every morphism s in the zigzag. By Lemma 4.8 we obtain the equality $G = F$ along this zigzag. In particular, we conclude that $G(b) = F(b)$. Now let x be an arbitrary object of $\mathbf{C}$. Since $\mathbf{C}$ is a finite poset, there exists a maximal object b and a morphism $x \to b$. Applying part (1) of Lemma 4.8 to this morphism and using the equality $G(b) = F(b)$, we obtain $G(x) = F(x)$. $\square$

Theorem 4.10. *Let p be an empirical model on an event scenario $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$ such that for every pair of maximal simplices $\sigma_1, \sigma_2 \in \Sigma$, there exists a zigzag in $\mathbf{C}_\Sigma$*

$$\sigma_1 \xleftarrow{s_1} \tau_1 \xrightarrow{s_2} \tau_2 \xleftarrow{s_3} \dots \xrightarrow{s_n} \sigma_2, \tag{18}$$

where $\eta_{\Sigma,F}(\kappa_F(p))(s_i)$ is an isomorphism for every morphism s_i appearing in (18) *(see Proposition 3.8). Then p is a vertex if and only if for any $G \in \mathsf{ESub}(F)$ satisfying $G \leq \eta_{\Sigma,F}(\kappa_F(p))$, there exists a maximal simplex $\sigma \in \Sigma$ such that $G(\sigma) = \eta_{\Sigma,F}(\kappa_F(p))(\sigma)$.*

Proof. Suppose first that p is a vertex. By Proposition 3.13, $\kappa_F(p)$ is minimal in $\mathsf{Emp}_{\mathbb{B}}(F)$. Hence, again by Proposition 3.13, $\eta_{\Sigma,F}(\kappa_F(p))$ is minimal in $\mathsf{ESub}(F)$. This implies that for every $G \in \mathsf{ESub}(F)$ such that $G \leq \eta_{\Sigma,F}(\kappa_F(p))$, we have $G = \eta_{\Sigma,F}(\kappa_F(p))$, and in particular the stated condition holds.

Conversely, assume that the stated condition holds. By Proposition 4.9, $\eta_{\Sigma,F}(\kappa_F(p))$ is minimal. Therefore, by Proposition 3.13, $\kappa_F(p)$ is minimal. It then follows from Corollary 4.6 that p is a vertex.

If p is a vertex then by Proposition 3.13 $\kappa_F(p)$ is minimal in $\mathsf{Emp}_{\mathbb{B}}(F)$. So by Proposition 3.13 $\eta_{\Sigma,F}(\kappa_F(p))$ is minimal in $\mathsf{ESub}(F)$, which means that for every $G \in \mathsf{ESub}(F)$ such that $G \leq \eta_{\Sigma,F}(\kappa_F(p))$, we have $G = F$. For the other direction, by Proposition 4.9 $\eta_{\Sigma,F}(\kappa_F(p))$ is minimal. Therefore, by Proposition 3.13 $\kappa_F(p)$ is minimal. As a result, using Corollary 4.6 we get that p is a vertex. $\square$

We now generalize the contextual vertex of Example 2.12 to the simplicial complex $\partial\Delta^n$. We begin with the case $n = 3$.

Example 4.11. Consider $T = (\partial\Delta^3, O_{\mathbb{Z}_2})$ (see Examples 2.1 and 2.5), where $V(\partial\Delta^3) = \{x, y, z, w\}$. We define $p \in \mathsf{Emp}(\mathcal{E}_T)$ as follows:

$$p^{\vec{a}}_{\{x,y,z\}} = \begin{cases} \frac{1}{3} & \text{if } \vec{a} \in \{(0,0,0),(1,1,0),(1,0,1)\} \\ 0 & \text{otherwise,} \end{cases}$$

$$p^{\vec{a}}_{\{x,y,w\}} = \begin{cases} \frac{1}{3} & \text{if } \vec{a} \in \{(0,0,0),(1,1,0),(1,0,1)\} \\ 0 & \text{otherwise,} \end{cases}$$

$$p^{\vec{a}}_{\{x,z,w\}} = \begin{cases} \frac{1}{3} & \text{if } \vec{a} \in \{(0,0,0),(1,1,0),(1,0,1)\} \\ 0 & \text{otherwise,} \end{cases}$$

$$p^{\vec{a}}_{\{y,z,w\}} = \begin{cases} \frac{1}{3} & \text{if } \vec{a} \in \{(1,0,0),(0,1,0),(0,0,1)\} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.12. *The empirical model p defined in Example 4.11 is a vertex.*

Proof. Let $F\colon \mathbf{C}^{\mathrm{op}}_{\partial\Delta^3} \to \mathbf{Set}$ be the proper event scenario $\eta_{\Sigma,\mathcal{E}_T}(\kappa_{\mathcal{E}_T}(p))$. We have

$$F(\{x,y,z\}) = F(\{x,y,w\}) = F(\{x,z,w\}) = \{(0,0,0),(1,1,0),(1,0,1)\},$$

$$\begin{gathered} F(\{y,z,w\}) = \{(1,0,0),(0,1,0),(0,0,1)\}, \\ F(\{y,z\}) = F(\{y,w\}) = F(\{z,w\}) = \{(0,0),(0,1),(1,0)\}, \\ F(\{x,y\}) = F(\{x,z\}) = F(\{x,w\}) = \{(0,0),(1,0),(1,1)\}, \\ F(\{x\}) = F(\{y\}) = F(\{z\}) = F(\{w\}) = \{0,1\} \end{gathered}$$

The first condition in Theorem 4.10 is clear. For the second codition, let $G \in \mathsf{ESub}(\mathcal{E}_T)$ such that $G \leq F$. Suppose $G(\{y,z,w\}) \neq \emptyset$. Without loss of generality, assume $(1,0,0) \in G(\{y,z,w\})$. Then $(0,0) \in G(\{z,w\})$, which implies $(0,0,0) \in G(\{x,z,w\})$. Similarly, we obtain $(0,0,0) \in G(\{x,y,z\})$ and $(0,0,0) \in G(\{x,y,w\})$. Therefore, we deduce $(0,0) \in G(\{y,w\})$ and $(0,0) \in G(\{y,z\})$, so both $(0,1,0)$ and $(0,0,1)$ belong to $G(\{y,z,w\})$. Hence, $G(\{y,z,w\}) = F(\{y,z,w\})$. By Theorem 4.10, we obtain that p is a vertex. □

Example 4.13. Let $T = (\partial\Delta^n, O_{\mathbb{Z}_2})$, where $x_1,\ldots,x_{n+1}$ are the vertices of $\partial\Delta^n$. We define $p \in \mathsf{Emp}(\mathcal{E}_T)$ as follows:

$$p^{\vec{a}}_{\{x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1}\}} = \begin{cases} \frac{1}{n} & \text{if } \vec{a} \in \{(0,\ldots,0),(1,1,0,\ldots,0),(1,0,1,0,\ldots,0),\ldots,(1,0,\ldots,0,1)\}, \\ 0 & \text{otherwise,} \end{cases}$$

for every $2 \leq j \leq n+1$.

$$p^{\vec{a}}_{\{x_2,\ldots,x_{n+1}\}} = \begin{cases} \frac{1}{n} & \text{if } \vec{a} \in \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.14. *The empirical model p of Example 4.13 is a vertex.*

Proof. The argument parallels the proof of Proposition 4.12. Let $F\colon \mathbf{C}^{\mathrm{op}}_{\partial\Delta^n} \to \mathbf{Set}$ be the proper event scenario $\eta_{\Sigma,\mathcal{E}_T}(\kappa_{\mathcal{E}_T}(p))$. The first condition in Theorem 4.10 is clear. For the second condition, let $G \in \mathsf{ESub}(\mathcal{E}_T)$ such that $G \leq F$. Since G is non-trivial, we have $G(\{x_2,\ldots,x_{n+1}\}) \neq \emptyset$. Assume

$$(1,0,\ldots,0) \in G(\{x_2,\ldots,x_{n+1}\}),$$

then $(0,\ldots,0) \in G(\{x_3,\ldots,x_{n+1}\})$. Since G is locally surjective, it follows that

$$(0,\ldots,0) \in G(\{x_1,x_3,\ldots,x_{n+1}\}).$$

Proceeding inductively, we deduce successively that $(0,\ldots,0) \in G(\{x_1,x_4\ldots,x_{n+1}\})$, so

$$(0,\ldots,0) \in G(\{x_1,x_2,x_4\ldots,x_{n+1}\}), \tag{19}$$

then we get that $(0,\ldots,0) \in G(\{x_2,x_4\ldots,x_{n+1}\})$, as a result

$$(0,1,0,\ldots,0) \in G(\{x_2,x_3,x_4\ldots,x_{n+1}\}).$$

Equation (19) implies also that $(0,\ldots,0) \in G(\{x_1,x_2,x_5\ldots,x_{n+1}\})$, so

$$(0,\ldots,0) \in G(\{x_1,x_2,x_3,x_5,\ldots,x_{n+1}\}), \tag{20}$$

then we get that $(0,\dots,0) \in G(\{x_2,x_3,x_5,\dots,x_{n+1}\})$, as a result

$$(0,0,1,0,\dots,0) \in G(\{x_2,x_3,x_4\dots,x_{n+1}\}).$$

Equation (20) implies also that $(0,\dots,0) \in G(\{x_1,x_2,x_3,x_6\dots,x_{n+1}\})$, so

$$(0,\dots,0) \in G(\{x_1,x_2,x_3,x_4,x_6,\dots,x_{n+1}\}), \tag{21}$$

then we obtain that $(0,\dots,0) \in G(\{x_2,x_3,x_4,x_6,\dots,x_{n+1}\})$, as a result

$$(0,0,0,1,0,\dots,0) \in G(\{x_2,x_3,x_4,x_5,\dots,x_{n+1}\}).$$

Continuing this process, we conclude that $G(\{x_2,x_3,x_4,x_5,\dots,x_{n+1}\}) = F(\{x_2,x_3,x_4,x_5,\dots,x_{n+1}\})$. Theorem 4.10 implies that p is a vertex. ☐

The simplicial complex $B(2,2)$ in Example 2.29 corresponds to the bipartite scenario in which each party has two measurements. We now generalize the vertex given in that example to scenarios with more parties, beginning with the case of three parties.

Example 4.15. Let $\Sigma = B(3,2)$ be the simplicial complex with the following maximal simplices:

$$\{x,y,z\},\ \{x,y,z'\},\ \{x,y',z\},\ \{x,y',z'\},\ \{x',y,z\},\ \{x',y,z'\},\ \{x',y',z\},\ \{x',y',z'\},$$

Let $T = (B(3,2), O_{\mathbb{Z}_2})$, we define an empirical model $p \in \mathsf{Emp}\,(\mathcal{E}_T)$ as follows:

$$p^{(a,b,c)}_{\{x,y,z\}} = \begin{cases} \frac{1}{4} & \text{if } a+b+c=1 \\ 0 & \text{otherwise,} \end{cases} \quad , \quad p^{(a,b,c)}_{A} = \begin{cases} \frac{1}{4} & \text{if } a+b+c=0 \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

for every maximal simplex $A \neq \{x,y,z\}$.

The empirical model above coincides with the vertex in [22, Equation (29)]. We prove that it is a vertex using Theorem 4.10.

Proposition 4.16. *The empirical model p defined in (22) is a vertex.*

Proof. Let $F\colon \mathbf{C}_\Sigma^{\mathrm{op}} \to \mathbf{Set}$ be the proper event scenario $\eta_{\Sigma,\mathcal{E}_T}(\kappa_{\mathcal{E}_T}(p))$. The first condition in Theorem 4.10 is clear. For the second condition, let $G \in \mathsf{ESub}(\mathcal{E}_T)$ such that $G \leq F$. Since G is non-trivial, assume that $(1,0,0) \in G(\{x,y,z\})$. Then $(0,0) \in G(\{y,z\})$. Because G is locally surjective, we obtain

$$(0,0,0) \in G(\{x',y,z\}).$$

By the same argument, we get

$$(0,0,0) \in G(\{x',y',z\}),\ (0,0,0) \in G(\{x,y',z\}),\ \text{and } (0,1,0) \in G(\{x,y,z\})$$

Consequently,

$$(0,1,1) \in G(\{x,y,z'\}),\ (0,1,1) \in G(\{x',y,z'\}),\ \text{and } (0,1,1) \in G(\{x',y,z\}),$$

and hence

$$(1,1,1) \in G(\{x,y,z\}).$$

On the other hand, we also have

$$(0,1,1) \in G(\{x',y',z\}),\ (0,1,1) \in G(\{x,y',z\}),\ \text{and } (0,0,1) \in G(\{x,y,z\}).$$

Altogether, this shows that

$$G(\{x,y,z\}) = F(\{x,y,z\}).$$

Note that for every maximal simplex A and every $x \in A$, we have $|F(A)| = |F(A \setminus \{x\})| = 4$, which implies that the map

$$F(A) \longrightarrow F(A \setminus \{x\})$$

is an isomorphism. Therefore, by Theorem 4.10 p is a vertex. $\square$

Example 4.17. Let $\Sigma = B(n,2)$ (Definition 2.2) be the simplicial complex whose maximal simplices are the sets consisting of one element from each of the following pairs:

$$\{x_1, x'_1\},\ \{x_2, x'_2\},\ \ldots,\ \{x_n, x'_n\}.$$

Let $T = (B(n,2), O_{\mathbb{Z}_2})$, we now define the simplicial distribution $p \in \mathsf{Emp}(\mathcal{E}_T)$ to be

$$p^{(a_1,\ldots,a_n)}_{\{x_1,\ldots,x_n\}} = \begin{cases} \dfrac{1}{2^{n-1}} & \text{if } \sum_{i=1}^n x_i = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad p^{(a_1,\ldots,a_n)}_{A} = \begin{cases} \dfrac{1}{2^{n-1}} & \text{if } \sum_{i=1}^n x_i = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

for every maximal simplex $A \neq \{x_1, \ldots, x_n\}$.

Proposition 4.18. *The empirical model p defined in (23) is a vertex.*

Proof. Let $F\colon \mathbf{C}^{\mathrm{op}}_\Sigma \to \mathbf{Set}$ be the proper event scenario $\eta_{\Sigma,\mathcal{E}_T}(\kappa_{\mathcal{E}_T}(p))$. The first condition in Theorem 4.10 is clear. For the second condition, let $G \in \mathsf{ESub}(\mathcal{E}_T)$ such that $G \leq F$. Since G is non-trivial, there exists $(i_1, i_2, \ldots, i_n) \in G(\{x_1, x_2, \ldots, x_n\})$. Take $(i_1, i_2, \ldots, i_n) \neq (j_1, j_2, \ldots, j_n) \in F(\{x_1, x_2, \ldots, x_n\})$. Since

$$i_1 + i_2 + \cdots + i_n = 1 \text{ and } j_1 + j_2 + \cdots + j_n = 1,$$

we have

$$(i_1 - j_1) + (i_2 - j_2) + \cdots + (i_n - j_n) = 0.$$

Thus, the number of indices k for which $i_k \neq j_k$ is even. Let $k < s$ be two such indices with $i_k \neq j_k$ and $i_s \neq j_s$. Because $(i_1, i_2, \ldots, i_n) \in G(\{x_1, x_2, \ldots, x_n\})$, we have

$$(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n) \in G(\{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\}).$$

Since G is locally surjective, $(i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_n) \notin G(\{x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n\})$, and $i_k \neq j_k$, it follows that

$$(i_1, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_n) \in G(\{x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n\}).$$

Consequently,

$$(i_1, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_{s-1}, i_{s+1}, \ldots, i_n) \in G(\{x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n\}).$$

By the same argument for the index s, we get

$$(i_1, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_{s-1}, i_s, i_{s+1}, \ldots, i_n) \in G(\{x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_{s-1}, x'_s, x_{s+1}, \ldots, x_n\})$$

and similarly,

$$(i_1, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_{s-1}, i_s, i_{s+1}, \ldots, i_n) \in G(\{x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{s-1}, x'_s, x_{s+1}, \ldots, x_n\})$$

and finally,

$$(i_1, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_{s-1}, j_s, i_{s+1}, \ldots, i_n) \in G(\{x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{s-1}, x_s, x_{s+1}, \ldots, x_n\})$$

Since the number of indices t with $i_t \neq j_t$ is even, repeating this process for all such indices yields

$$(j_1, j_2, \ldots, j_n) \in G(\{x_1, x_2, \ldots, x_n\}).$$

We proved that $G(\{x_1, x_2, \ldots, x_n\}) = F(\{x_1, x_2, \ldots, x_n\})$. Theorem 4.10 implies that p is a vertex. □

5 Characterization of possibilistic simplicial distributions

In this section, we give a structural characterization of possibilistic simplicial distributions. We show that they are equivalently described as sub-proper bundle scenarios, via a functorial construction based on the relative Grothendieck construction. This correspondence allows us to translate probabilistic support conditions into purely simplicial data.

5.1 The functor of sub-bundle scenarios

In this section, we define the functor of sub-proper bundle scenarios (or shortly sub-bundle scenarios) using the relative Grothendieck construction.

Lemma 5.1. *Given Diagram (7), where α is surjective and f is a proper bundle scenario. Then g is also a proper bundle scenario.*

Proof. Given a commutative diagram:

$$\begin{tikzcd} \Delta[n] \arrow[r, "e"] \arrow[d, "\theta^*"'] & F \arrow[d, "g"] \\ \Delta[m] \arrow[r, "x"] & X \end{tikzcd} \tag{24}$$

Since α_n is surjective there exist $e' \in E_n$ such that $\alpha_n(e') = e$. Then, we have

$$x \circ \theta^* = g_n(e) = g_n(\alpha_n(e')) = f_n \circ e'$$

f is a proper bundle scenario, so there exists $\tilde{e} \in E_m$ such that the following diagram commutes:

$$\begin{tikzcd} \Delta[n] \arrow[r, "e'"] \arrow[d, "\theta^*"'] & E \arrow[d, "f"] \\ \Delta[m] \arrow[r, "x"] \arrow[ur, "\tilde{e}"] & X \end{tikzcd} \tag{25}$$

Composing Diagrams (25) and (7) gives us a lifting for Diagram (24). □

Definition 5.2. Given a simplicial set X, we define the functor $\mathsf{Sub}(X) \colon \mathsf{sBund}(X) \to \mathbf{Set}$ by

- sending a bundle scenario $f \colon E \to X$ in $\mathsf{sBund}(X)$ to the set

$$\mathsf{Sub}(X)(f) = \{E' \in \mathbf{sSet} : E' \subset E \text{ and } f|_{E'} \text{ is a proper bundle scenario}\}$$

- sending a morphism $\alpha\colon f \to g$ to the set map $\mathsf{Sub}(X)(\alpha)\colon \mathsf{Sub}(X)(f) \to \mathsf{Sub}(X)(g)$ that sends $E' \in \mathsf{Sub}(X)(f)$ to $\alpha(E')$.

Note that $\alpha(E') \in \mathsf{Sub}(X)(g)$ by Lemma 5.1.

Definition 5.3. For a simplicial set map $\pi\colon Y \to X$, we define a morphism of $\mathbf{Cat}/\!\!/\mathbf{Set}$ from $\mathsf{Sub}(X)$ to $\mathsf{Sub}(Y)$, which consists of the functor $\pi^*\colon \mathsf{sBund}(X) \to \mathsf{sBund}(Y)$ and the natural transformation that is determined by

$$\mathsf{Sub}(\pi)_f\colon \mathsf{Sub}(X)(f) \to \mathsf{Sub}(Y)(\pi^*(f))$$

that sends $E' \in \mathsf{Sub}(X)(f)$ to $E' \times_X Y \in \mathsf{Sub}(Y)(\pi^*(f))$. See Diagram (26) and note that $\pi^*(f|_{E'}) = \pi^*(f)|_{E'\times_X Y}$:

$$\begin{tikzcd}
E' \arrow[d, hook] & E' \times_X Y \arrow[l] \arrow[dd, "\pi^*(f|_{E'})"] \\
E \arrow[d, "f"] & \\
X & Y \arrow[l, "\pi"]
\end{tikzcd} \tag{26}$$

We defined the morphism $(\pi^*, \mathsf{Sub}(\pi)_-)$ as follows:

$$\begin{tikzcd}
\mathsf{sBund}(X) \arrow[rr, "\pi^*"] \arrow[dr, "\mathsf{Sub}(X)"', ""{name=L}] & & \mathsf{sBund}(Y) \arrow[dl, "\mathsf{Sub}(Y)", ""'{name=R}] \\
& \mathbf{Set} &
\arrow[Rightarrow, from=L, to=R, "\mathsf{Sub}(\pi)_-"]
\end{tikzcd}$$

Thus, we have a functor

$$\mathsf{Sub}_-\colon \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Cat}/\!\!/\mathbf{Set}$$

Definition 5.4. The *sub-proper bundle scenarios functor* or shortly *sub-bundle scenarios functor* is defined to be the relative Grothendieck construction of Sub_-:

$$\mathsf{Sub} = \int_{\mathbf{sSet}^{\mathrm{op}}} \mathsf{Sub}_-\colon \mathbf{sScen} \to \mathbf{Set}.$$

5.2 Possibilistic simplicial distribution as sub-bundle scenarios

In this section, we show that the functor of sub-bundle scenarios is equivalent to the functor of possibilistic simplicial distributions defined in Section 2.4.

Definition 5.5. For a simplicial bundle scenario $f\colon E \to X$, we define the map

$$\zeta_{X,f}\colon \mathsf{sDist}_{\mathbb{B}}(X)(f) \to \mathsf{Sub}(X)(f)$$

by sending $p \in \mathsf{sDist}_{\mathbb{B}}(X)(f)$ to the simplicial set $\zeta_{X,f}(p)$ given by

$$\zeta_{X,f}(p)_n = \{e \in E_n \mid p_n(f_n(e))(e) \neq 0\} \tag{27}$$

Proposition 5.6. *The maps $\zeta_{X,f}$ from Definition 5.5 form a natural isomorphism ζ_X from $\mathsf{sDist}_{\mathbb{B}}(X)$ to $\mathsf{Sub}(X)$.*

Proof. First, we prove that $\zeta_{X,f}(p)$ is a simplicial subset of E. Let $e \in \zeta_{X,f}(p)_n$ and $0 \le i \le n$. Then

$$\begin{aligned} p_{n-1}(f_{n-1}(d_i(e)))(d_i(e)) &= D_{\mathbb{B}}(d_i)\big(p_n(f_n(e))\big)(d_i(e)) \\ &= \sum_{e' \,:\, d_i(e')=d_i(e)} p_n(f_n(e))(e') \\ &= p_n(f_n(e))(e) + \cdots \neq 0, \end{aligned}$$

so $d_i(e) \in \zeta_{X,f}(p)_{n-1}$. Similarly, we obtain that $s_i(e) \in \zeta_{X,f}(p)_{n+1}$. Now we prove that $\zeta_{X,f}(p)$ lies in $\mathsf{Sub}(X)(f)$:

Surjectivity: Let $x \in X_n$. There exists $e \in E_n$ such that $p_n(x)(e) \neq 0$. By Proposition 2.23 $f_n(e) = x$, so $e \in \zeta_{X,f}(p)_n$.

Local surjectivity: Let $e \in \zeta_{X,f}(p)_{n-1}$, $x \in X_n$, and $0 \le i \le n$ such that $f_n(e) = d_i(x)$. Then

$$\sum_{e' \,:\, d_i(e')=e} p_n(x)(e') = D_{\mathbb{B}}(d_i)(p_n(x))(e) = p_{n-1}(d_i(x))(e) = p_n(f_n(e))(e) \neq 0.$$

Thus there exists $e' \in E_n$ with $d_i(e') = e$ and $p_n(x)(e') \neq 0$. By Proposition 2.23, $x = f_n(e')$, so $e' \in \zeta_{X,f}(p)_n$.

Discreteness over vertices follows by a similar argument.

To prove the naturality of ζ_X, given $\alpha\colon f \to g$ in $\mathsf{sBund}(X)$ as in (7), we prove that the following diagram commutes:

$$\begin{tikzcd} \mathsf{sDist}_{\mathbb{B}}(X)(f) \arrow[r, "\zeta_{X,f}"] \arrow[d, "\mathsf{sDist}_{\mathbb{B}}(X)(\alpha)"'] & \mathsf{Sub}(X)(f) \arrow[d, "\mathsf{Sub}(X)(\alpha)"] \\ \mathsf{sDist}_{\mathbb{B}}(X)(g) \arrow[r, "\zeta_{X,g}"'] & \mathsf{Sub}(X)(g) \end{tikzcd}$$

Given $p \in \mathsf{sDist}_{\mathbb{B}}(X)(f)$, we have

$$\begin{aligned} \zeta_{X,g}(D_{\mathbb{B}}(\alpha) \circ p)_n &= \{e \in F_n \mid (D_{\mathbb{B}}(\alpha_n) \circ p_n)(g_n(e))(e) \neq 0\} \\ &= \{e \in F_n \mid \sum_{\alpha_n(e')=e} p_n(g_n(e))(e') \neq 0\} \\ &= \{e \in F_n \mid \exists e' \in E_n \text{ s.t } \alpha_n(e') = e \text{ and } p_n(g_n(e))(e') \neq 0\} \\ &= \{e \in F_n \mid \exists e' \in E_n \text{ s.t } \alpha_n(e') = e \text{ and } p_n(g_n(\alpha_n(e')))(e') \neq 0\} \\ &= \{e \in F_n \mid \exists e' \in E_n \text{ s.t } \alpha_n(e') = e \text{ and } p_n(f_n(e'))(e') \neq 0\} \\ &= \alpha_n\big(\{e' \in E_n \mid p_n(f_n(e'))(e') \neq 0\}\big) \\ &= \alpha_n(\zeta_{X,f}(p)_n). \end{aligned}$$

Finally, we prove that $\zeta_{X,f}$ is an isomorphism. Given $E' \in \mathsf{Sub}(X)(f)$, define $S_f(E') \in \mathsf{sDist}_{\mathbb{B}}(X)(f)$ by

$$S_f(E')_n(x)(e) = \begin{cases} 1 & \text{if } e \in E'_n \text{ and } f_n(e) = x, \\ 0 & \text{otherwise.} \end{cases}$$

We prove that $S_f(E') \in \mathsf{sDist}_{\mathbb{B}}(X)(f)$. Given $x \in X_n$. Since $f_n|_{E'_n}\colon E'_n \to X_n$ is surjective, there is $e \in E'_n$ with $f_n(e) = x$, hence $S_f(E')_n(x)(e) = 1$. Recall also that, by Definition 2.17, we have $|E_n| < \infty$. Thus $S_f(E')_n(x) \in D_{\mathbb{B}}(E_n)$. Next, we prove the simpliciality of $S_f(E')$. For $x \in X_n$, $e \in E_{n-1}$, by definition we have $S_f(E')_{n-1}(d_i^X(x))(e) = 1$ if and only if $e \in E'_{n-1}$ and $f_{n-1}(e) = d_i^X(x)$. This is equivalent to commutativity of

$$\begin{tikzcd}
\Delta[n-1] \arrow[r, "e"] \arrow[d, "d^i"'] & E' \arrow[d, "f|_{E'}"] \\
\Delta[n] \arrow[r, "x"] & X
\end{tikzcd}$$

On the other hand, $D_{\mathbb{B}}(d_i^E)(S_f(E')_n(x))(e) = \sum_{e' \colon d_i^E(e')=e} S_f(E')_n(x)(e')$ is equal to 1 if and only if there exists $e' \in E_n$ that makes the following two triangles commute

$$\begin{tikzcd}
\Delta[n-1] \arrow[r, "e"] \arrow[d, "d^i"'] & E' \arrow[d, "f|_{E'}"] \\
\Delta[n] \arrow[r, "x"] \arrow[ur, "e'"] & X
\end{tikzcd}$$

Since $f|_{E'}$ is locally surjective, we obtain

$$S_f(E')_{n-1}(d_i^X(x)) = D_{\mathbb{B}}(d_i^E)\big(S_f(E')_n(x)\big).$$

Similarly, using the fact that $f|_{E'}$ is discrete over vertices, we get

$$S_f(E')_{n+1}(s_i^X(x)) = D_{\mathbb{B}}(s_i^E)(S_f(E')_n(x)).$$

Finally, we prove that $D_{\mathbb{B}}(f) \circ S_f(E') = \delta_X$. Given $x, x' \in X_n$, then

$$D_{\mathbb{B}}(f_n)(S_f(E')_n(x))(x') = \sum_{e \colon f_n(e)=x'} S_f(E')_n(x)(e) = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases}$$

Now let $p \in \mathsf{sDist}_{\mathbb{B}}(X)(f)$, $x \in X_n$, and $e \in f_n^{-1}(x)$. Then $S_f(\zeta_{X,f}(p))_n(x)(e) = 1$ if and only if $e \in \zeta_{X,f}(p)_n$, which holds if and only if $p_n(x)(e) = 1$. Hence,

$$S_f(\zeta_{X,f}(p)) = p$$

Conversely, if $E' \in \mathsf{Sub}(X)(f)$, then $e \in \zeta_{X,f}(S_f(E'))_n$ if and only if $S_f(E')(f_n(e))(e) = 1$, which holds if and only if $e \in E'_n$. We also proved that

$$\zeta_{X,f}(S_f(E')) = E'$$

We therefore conclude that $\zeta_{X,f}$ is an isomorphism. □

Proposition 5.7. *ζ is a natural isomorphism from $\mathsf{sDist}_{\mathbb{B},-}$ to Sub_-.*

Proof. By Proposition 5.6, for every object $X \in \mathbf{sSet}$, we have the isomorphism $(\mathrm{id}_{\mathsf{sBund}(X)}, \zeta_X)$ in $\mathbf{Cat}/\!\!/\mathbf{sSet}$:

$$\begin{tikzcd}
\mathsf{sBund}(X) \arrow[rr, equal] \arrow[dr, "\mathsf{sDist}_{\mathbb{B}}(X)"', ""{name=A}] & & \mathsf{sBund}(X) \arrow[dl, "\mathsf{Sub}(X)", ""'{name=B}] \\
& \mathbf{Set} &
\arrow[from=A, to=B, Rightarrow, "\zeta_X"]
\end{tikzcd}$$

We should prove that the following diagram commutes in the category $\mathbf{Cat}/\!\!/\mathbf{sSet}$:

$$\begin{tikzcd}
\mathsf{sDist}_{\mathbb{B}}(X) \arrow[r, "{(\pi^*,\pi^*_-)}"] \arrow[d, "{(\mathrm{id}_{\mathsf{sBund}(X)},\zeta_X)}"'] & \mathsf{sDist}_{\mathbb{B}}(Y) \arrow[d, "{(\mathrm{id}_{\mathsf{sBund}(Y)},\zeta_Y)}"] \\
\mathsf{Sub}(X) \arrow[r, "{(\pi^*,\mathsf{Sub}(\pi)_-)}"'] & \mathsf{Sub}(Y)
\end{tikzcd}$$

That means, to prove that for every simplicial bundle scenario $f\colon E \to X$ the following diagram commutes:

$$\begin{tikzcd}
\mathsf{sDist}_{\mathbb{B}}(X)(f) \arrow[r, "\pi^*_f"] \arrow[d, "\zeta_{X,f}"'] & \mathsf{sDist}_{\mathbb{B}}(Y)(\pi^*(f)) \arrow[d, "\zeta_{Y,\pi^*(f)}"] \\
\mathsf{Sub}(X)(f) \arrow[r, "\mathsf{Sub}(\pi)_f"'] & \mathsf{Sub}(Y)(\pi^*(f))
\end{tikzcd}$$

Given a simplicial distribution $p \in \mathsf{sDist}_{\mathbb{B}}(X)(f)$, we have

$$\begin{aligned}
\zeta_{Y,\pi^*(f)}\left(\pi^*_f(p)\right)_n &= \{(e,y) \in E_n \times_{X_n} Y_n :\ \pi^*_f(p)(y)(e,y) \neq 0\} \\
&= \{(e,y) \in E_n \times_{X_n} Y_n :\ p_n(\pi_n(y))(e) \neq 0\} \\
&= \{(e,y) \in E_n \times Y_n :\ f_n(e) = \pi_n(y) \text{ and } p_n(f_n(e))(e) \neq 0\} \\
&= \{(e,y) \in E_n \times Y_n :\ f_n(e) = \pi_n(y) \text{ and } e \in \zeta_{X,f}(p)_n\} \\
&= \zeta_{X,f}(p)_n \times_{X_n} Y_n \\
&= \mathsf{Sub}(\pi)_f\left(\zeta_{X,f}(p)\right)_n
\end{aligned}$$

see Equation (9). ☐

Theorem 5.8. *The functors* $\mathsf{sDist}_{\mathbb{B}}\colon \mathbf{sScen} \to \mathbf{Set}$ *and* $\mathsf{Sub}\colon \mathbf{sScen} \to \mathbf{Set}$ *are equivalent.*

Proof. Directly by Propositions 5.7 and A.4. ☐

6 Geometrical characterization of extremal simplicial distributions

In this section, we use the characterization of possibilitic simplicial distributions established in the previous section, in order to give geometric properties of proper bundle scenarios that characterizes a family of extremal simplicial distributions.

We begin by adapting Proposition 4.4 to the setting of simplicial distributions. This simplicial version is also a special case of [20, part (3) of Corollary 5.5].

For a simplicial bundle scenario f*, the projection in (5) induces a projection*

$$\kappa_f\colon \mathsf{sDist}(f) \to \mathsf{sDist}_{\mathbb{B}}(f).$$

Proposition 6.1. *Given a simplicial bundle scenario* f*. A simplicial distribution* $p \in \mathsf{sDist}(f)$ *is a vertex if and only if* $\kappa_f^{-1}(\kappa_f(p)) = \{p\}$*.*

Now we use the results of Section 5.2 to obtain a family of extremal simplicial distributions.

Definition 6.2. Let $f\colon E \to X$ be a proper bundle scenario. For $e_1 \in E_{n_1}$ and $e_2 \in E_{n_2}$, we write $e_1 \leftrightarrow e_2$ if there exist ordinal maps θ_1 and θ_2 such that $\theta_1^*(e_1) = \theta_2^*(e_2)$, and the following squares admit unique liftings:

$$\begin{tikzcd}
\Delta[n_1 - k_1] \arrow[r, "\theta_1^*(e_1)"] \arrow[d, hook, "\theta_1"] & E \arrow[d, "f"] \\
\Delta[n_1] \arrow[r, "f_{n_1}(e_1)"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta[n_2 - k_2] \arrow[r, "\theta_2^*(e_2)"] \arrow[d, hook, "\theta_2"] & E \arrow[d, "f"] \\
\Delta[n_2] \arrow[r, "f_{n_2}(e_2)"] & X
\end{tikzcd}$$

In fact, the unique liftings are given by the simplicial maps

$$\Delta[n_1] \xrightarrow{e_1} E \quad \text{and} \quad \Delta[n_2] \xrightarrow{e_2} E.$$

We say that e_1 and e_2 are *strongly connected by* f if there exist simplices $e_1', \dots, e_m'$ such that

$$e_1 \leftrightarrow e_1' \leftrightarrow \cdots \leftrightarrow e_m' \leftrightarrow e_2.$$

Note that strong connectivity defines an equivalence relation, which we denote by $\sim_f$.

Proposition 6.3. *Let p be a simplicial distribution on a simplicial bundle scenario $f\colon E \to X$, and let $E' := \zeta_{X,f}(\kappa_f(p))$ (see Equation (27)). Given $e_1 \in E'_{n_1}$ and $e_2 \in E'_{n_2}$, if $e_1 \sim_{f|_{E'}} e_2$, then*

$$p_{f_{n_1}(e_1)}(e_1) = p_{f_{n_2}(e_2)}(e_2).$$

Proof. It suffices to prove the claim in the case $e_1 \leftrightarrow e_2$. Suppose there exist ordinal maps θ_1 and θ_2 such that $\theta_1^*(e_1) = \theta_2^*(e_2)$, and the following squares admit unique liftings:

$$\begin{tikzcd}
\Delta[n_1 - k_1] \arrow[r, "\theta_1^*(e_1)"] \arrow[d, hook, "\theta_1"] & E' \arrow[d, "f|_{E'}"] \\
\Delta[n_1] \arrow[r, "f_{n_1}(e_1)"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta[n_2 - k_2] \arrow[r, "\theta_2^*(e_2)"] \arrow[d, hook, "\theta_2"] & E' \arrow[d, "f|_{E'}"] \\
\Delta[n_2] \arrow[r, "f_{n_2}(e_2)"] & X
\end{tikzcd}$$

Denote $\sigma_i := f_{n_i}(e_i)$. By the uniqueness of lifting in the left square and since $\zeta_{X,f}(\kappa_f(p)) = E'$, we obtain

$$p_{\theta_1^*(\sigma_1)}(\theta_1^*(e_1)) = D(\theta_1^*)(p_{\sigma_1})(\theta_1^*(e_1)) = \sum_{\substack{e' \in E'_{n_1} \\ \theta_1^*(e') = \theta_1^*(e) \\ f_{n_1}(e') = \sigma_1}} p_{\sigma_1}(e') = p_{\sigma_1}(e_1).$$

Similarly, we have $p_{\theta_2^*(\sigma_2)}(\theta_2^*(e_2)) = p_{\sigma_2}(e_2)$. Note that $\theta_1^*(e_1) = \theta_2^*(e_2)$ implies that

$$\theta_1^*(\sigma_1) = \theta_1^*(f_{n_1}(e_1)) = \theta_2^*(f_{n_2}(e_2)) = \theta_2^*(\sigma_2)$$

We conclude that $p_{\sigma_1}(e_1) = p_{\sigma_2}(e_2)$. ☐

Corollary 6.4. *Let p be a simplicial distribution on a simplicial bundle scenario $f\colon E \to X$. If every pair of generator simplices of $\zeta_{X,f}(\kappa_f(p))$ is strongly connected by $g := f|_{\zeta_{X,f}(\kappa_f(p))}$, then p is a vertex of $\mathsf{sDist}(f)$.*

Proof. By Proposition 6.3 and Corollary 2.24, there exists a unique value t such that $p_{f_n(e)}(e) = t$ whenever $p_{f_n(e)}(e) \neq 0$. For a generator simplex $\sigma \in X_n$ and $e \in f_n^{-1}(\sigma)$, the simplex e is also a generator. Since

$$\sum_{e \in f_n^{-1}(\sigma)} p_\sigma(e) = \sum_{e \in g_n^{-1}(\sigma)} p_\sigma(e) = 1,$$

we conclude that $t = \frac{1}{|g_n^{-1}(\sigma)|}$. Thus, p is uniquely determined by its Boolean shadow under κ_f. Hence, by Proposition 6.1, p is a vertex of $\mathsf{sDist}(f)$. $\square$

Definition 6.5. The *k-cyclic bundle scenario* over the circle $C^{(n)}$ is the simplicial map

$$f = f^{n,k} \colon C^{(nk)} \to C^{(n)},$$

defined by setting $f_{\sigma_j} = \sigma_{[j]}$, where $[j]$ denotes the residue class of j modulo n.

Note that every pair of edges in $C^{(nk)}$ is strongly connected by $f^{n,k}$. Let p be a k-order cycle distribution on $f_{C^{(n)},m} \colon C^{(n)} \times \Delta_{\mathbb{Z}_m} \to C^{(n)}$ (see Example 2.28). Then the restricted scenario

$$f_{C^{(n)},m}\big|_{\zeta_{X,f}(\kappa_f(p))}$$

is isomorphic to the k-cyclic scenario $f^{n,k}$. Consequently, by Corollary 6.4, every k-order cycle distribution is a vertex of $\mathsf{sDist}(f_{C^{(n)},m})$.

Next, We illustrate the geometric shape of the vertex defined in (22).

Example 6.6. Consider the bundle scenario $f = f_{sB(3,2),2}$ (see Definition 2.14 and example 2.18), where $A_1 = \{x, x'\}$, $A_2 = \{y, y'\}$, and $A_3 = \{z, z'\}$. We define a simplicial distribution $p \in \mathsf{sDist}(f)$ by

$$p_{\{x,y,z\}}^{(\{x,y,z\},(a,b,c))} = \begin{cases} \frac{1}{4} & \text{if } \ a+b+c=1 \\ 0 & \text{otherwise,} \end{cases} \quad , \quad p_\sigma^{(\sigma,(a,b,c))} = \begin{cases} \frac{1}{4} & \text{if } \ a+b+c=0 \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

for every generating simplex $\sigma \neq \{x, y, z\}$. From Figures 1 and 2, one sees that every pair of triangles in $\zeta_{sB(3,2),f}(\kappa_f(p))$ is strongly connected by $f|_{\zeta_{sB(3,2),f}(\kappa_f(p))}$. Therefore, by Corollary 6.4, p is a vertex.

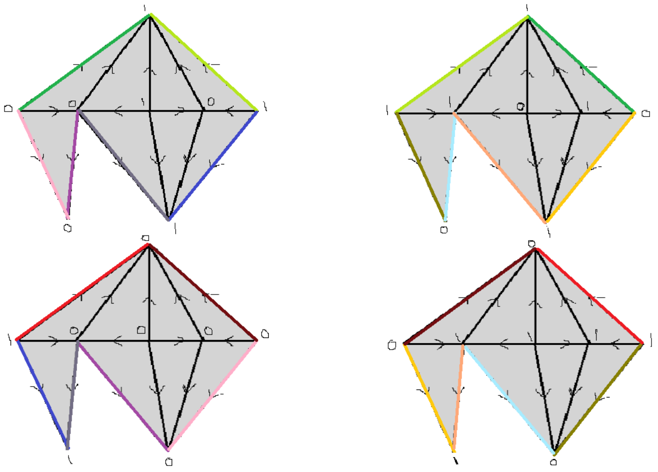

Figure 1: The geometric shape of $\zeta_{sB(3,2),f}\left(\kappa_f(p)\right)$ in Example 6.6

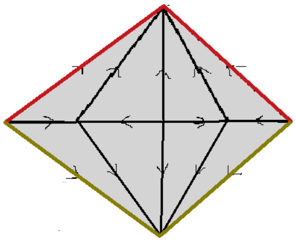

Figure 2: The geometric shape of $sB(3,2)$

As mentioned earlier, the vertex in Example 6.6 appears in [22, Equation (29)]. In fact, it is identified there as one of the three classes of contextual vertices on the $(3,2,2)$ *Bell scenario, which are referred to as* three-way nonlocal vertices*. It turns out that the other two vertices in this classification also satisfy the condition of Corollary 6.4; see Figures 3 and 4.*

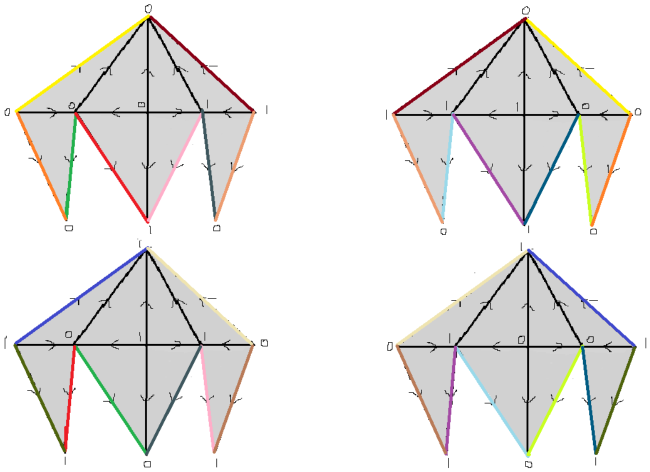

Figure 3: The geometric shape of the Boolean shadow of the vertex [22, Equation (27)]

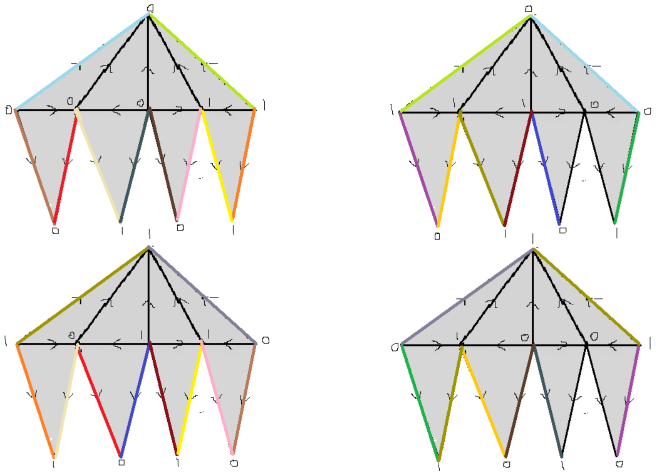

Figure 4: The geometric shape of the Boolean shadow of the vertex [22, Equation (28)]

We conclude this section with a simpler example.

Example 6.7. Let X be the simplicial set generated by four 2-simplices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, glued along faces as follows:

$$d_0(\sigma_1) = d_0(\sigma_2),\ d_2(\sigma_2) = d_2(\sigma_3),\ d_0(\sigma_3) = d_0(\sigma_4),\ d_2(\sigma_4) = d_2(\sigma_1),$$

See Figure 6. We define a simplicial distribution p on the simplicial bundle scenario

$$f = f_{X,2} \colon X \times \Delta_{\mathbb{Z}_4} \to X$$

by

$$p_{\sigma_i}^{(\sigma_i,(a,b,c))} = \begin{cases} \frac{1}{4} & \text{if } (a,b,c) \in A \\ 0 & \text{otherwise,} \end{cases} \quad , \quad p_{\sigma_3}^{(\sigma_3,(a,b,c))} = \begin{cases} \frac{1}{4} & \text{if } (a,b,c) \in B \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in \{1, 2, 4\}$ and

$$A = \{(0,0,0), (1,0,1), (2,0,2), (3,0,3)\}, \qquad B = \{(0,0,3), (1,0,0), (2,0,1), (3,0,2)\}$$

From the geometric shape of $\zeta_{X,f}(\kappa_f(p))$ (see Figure 5), one sees that p satisfies the condition of Corollary 6.4. Therefore, p is a vertex of $\mathsf{sDist}(f)$. In fact, $\zeta_{X,f}(\kappa_f(p))$ is a disc composed of 16 tringles.

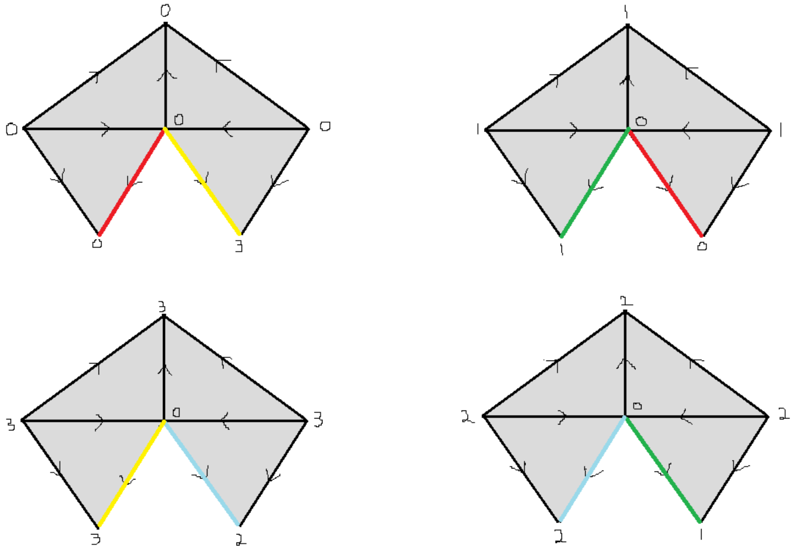

Figure 5: The geometric shape of $\zeta_{X,f}\left(\kappa_f(p)\right)$ in Example 6.7

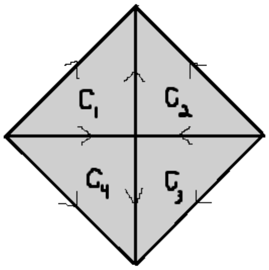

Figure 6: Base space of glued four triangles

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A The Grothendieck and the relative Grothendieck constructions

This section recalls the Grothendieck construction [15] and presents a relative version that extends to 2*-categories [15, Section XII.3].*

Definition A.1. Let $\mathbf{C}$ be a category, and let $\mathbf{Cat}$ denote the category of locally small categories. Given a functor $F\colon \mathbf{C} \to \mathbf{Cat}$, its Grothendieck construction is the category $\int_{\mathbf{C}} F$ defined as follows:

- The objects are pairs (c, x) with $c \in \mathbf{C}$ and $x \in F(c)$.
- A morphism $(c, x) \to (d, y)$ consists of a morphism $h\colon c \to d$ in $\mathbf{C}$ together with a morphism $\gamma\colon F(h)(x) \to y$ in $F(d)$.

The Grothendieck construction is functorial, that meanes it is a functor from $\mathsf{Fun}(\mathbf{C}, \mathbf{Cat})$ *to* $\mathbf{Cat}$*. In addition, there is a canonical projection*

$$\int_{\mathbf{C}} F \to \mathbf{C}.$$

which forgets the second component. There is also a contravariant version of the Grothendieck construction for a functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$*. The objects are again pairs* (c, x)*, while a morphism* $(c, x) \to (d, y)$ *now consists of a morphism* $h\colon d \to c$ *in* $\mathbf{C}$ *and a map* $\gamma\colon F(h)(x) \to y$ *in* $F(d)$*.*

We next introduce a version of the Grothendieck construction adapted to 2-categorical settings. A 2-category consists of:

- *objects,*
- *1-morphisms between objects,*
- *2-morphisms between 1-morphisms.*

Our example is **Cat***, whose 1-morphisms are functors, and whose 2-morphisms are natural transformations.*

Definition A.2. Let $\mathcal{K}$ be a 2-category and c an object of $\mathcal{K}$. The *thick slice category $\mathcal{K}/\!\!/c$* is defined as follows:

- Objects are morphisms $f\colon a \to b$ in $\mathcal{K}$.
- A morphism from $f\colon a \to c$ to $g\colon b \to c$ is a pair (h, η) where $h\colon a \to b$ is a 1-morphism in $\mathcal{K}$ and $\eta\colon f \Rightarrow g \circ h$ is a 2-morphism; see Diagram (29).

$$\begin{tikzcd}
a \arrow[rrrr, "h"] \arrow[ddrr, "f"'] & & & & b \arrow[ddll, "g"] \\
& {} \arrow[rr, Rightarrow, "\eta"] & & {} & \\
& & c & &
\end{tikzcd} \tag{29}$$

We define the functor $\Pi\colon \mathcal{K}/\!\!/c \to \mathcal{K}$ that sends $f\colon a \to b$ to a and (h, η) to h.

Definition A.3. Let $\mathbf{C}$ and $\mathbf{E}$ be categories, and let $F\colon \mathbf{C} \to \mathbf{Cat}/\!\!/\mathbf{E}$ be a functor. Write $\bar{F}$ for the composite $\Pi \circ F\colon \mathbf{C} \to \mathbf{Cat}$. The *relative Grothendieck construction* is the functor

$$\int_{\mathbf{C}} F\colon \int_{\mathbf{C}} \bar{F} \to \mathbf{E},$$

defined as follows:

- For an object (c, x) of $\int_{\mathbf{C}} \bar{F}$, where x is an object of $\bar{F}(c)$, set $(\int_{\mathbf{C}} F)(c, x) = F(c)(x)$.
- For a morphism $(h, \gamma)\colon (c, x) \to (d, y)$ in $\int_{\mathbf{C}} \bar{F}$, where $\gamma\colon \bar{F}(h)(x) \to y$ lies in $\bar{F}(d)$. If $F(h) = (\bar{F}(h), \eta)$, then we define
$$(\int_{\mathbf{C}} F)(h, \gamma) = F(d)(\gamma) \circ \eta_x.$$

The composite above has the form

$$F(c)(x) \xrightarrow{\eta_x} F(d)(\bar{F}(h)(x)) \xrightarrow{F(d)(\gamma)} F(d)(y).$$

where η is a natural transformation $F(c) \to F(d) \circ \bar{F}(h)$, and $F(d)$ is a functor $\bar{F}(d) \to E$.

The relative Grothendieck construction is functorial. Means, it is a functor

$$\mathsf{Fun}(\mathbf{C}, \mathbf{Cat}/\!\!/\mathbf{E}) \longrightarrow \mathbf{Cat}/\!\!/\mathbf{E}.$$

As a consequence, we obtain the following result.

Proposition A.4. *A natural isomorphism between functors*

$$F\colon \mathbf{C} \longrightarrow \mathbf{Cat}/\!\!/\mathbf{E} \quad \text{and} \quad G\colon \mathbf{C} \longrightarrow \mathbf{Cat}/\!\!/\mathbf{E}$$

induces a natural isomorphism between the corresponding relative Grothendieck functors

$$\int_{\mathbf{C}} F \quad \text{and} \quad \int_{\mathbf{C}} G.$$