Combinatory Completeness in Structured Multicategories

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A *combinatory algebra* is an algebraic model of combinatory logic [1]. One defines an *applicative system* to be a set A of *combinators* together with a (total) function $\bullet : A \times A \to A$ called *application*. Adopting the usual conventions, we treat application as a left-associative infix binary operation whose symbol is often omitted, so that $xy = x \bullet y = \bullet(x,y)$ and $xyz = (xy)z$. Next, we define:

- a B combinator to be some $\mathsf{B} \in A$ such that $\mathsf{B}xyz = x(yz)$ for all $x,y,z \in A$.
- a C combinator to be some $\mathsf{C} \in A$ such that $\mathsf{C}xyz = xzy$ for all $x,y,z \in A$.
- a K combinator to be some $\mathsf{K} \in A$ such that $\mathsf{K}xy = x$ for all $x,y \in A$.
- a W combinator to be some $\mathsf{W} \in A$ such that $\mathsf{W}xy = xyy$ for all $x,y \in A$.
- an I combinator to be some $\mathsf{I} \in A$ such that $\mathsf{I}x = x$ for all $x \in A$.

We call applicative systems which have some subset H of these combinators H-algebras, so that for example a BI-algebra is an applicative system with a B and I combinator. Then in particular a combinatory algebra is defined to be a BCKWI-algebra.

Given an applicative system $(A,\bullet)$ one defines a *polynomial* in variables $x_1,..,x_n$ to be one of: a variable x_i where $1 \leq i \leq n$; a constant $a \in A$; or $t \bullet s$ where t and s are polynomials in $x_1,..,x_n$. A polynomial t is said to be *computable* in case there exists $a \in A$ such that for all $b_1,..,b_n \in A$ we have:

$$ab_1 \cdots b_n = t[b_1,..,b_n/x_1,..,x_n]$$

An applicative system in which every polynomial is computable is said to be *combinatory complete*. This turns out to be equivalent to being a combinatory algebra, as in:

Theorem 1 (After [2], Chapter 6)**.** *An applicative system $(A,\bullet)$ is combinatory complete if and only if it is a combinatory algebra.*

In fact, to obtain a combinatory algebra it is enough to ask that only the *regular* polynomials are computable, where a polynomial is said to be regular in case it contains no constants. For many — but not all — classes of polynomial, for the regular polynomials to be computable is equivalent to all of the polynomials being computable.

A natural question is whether there exist analogues of Theorem 1 characterising applicative systems in which only some subset of the distinguished elements of a combinatory algebra need exist. For example, it is known that an applicative system is a BCI-algebra if and only if every *linear* polynomial is computable, where a polynomial t in variables $x_1,..,x_n$ is said to be linear in case each variable $x_1,..,x_n$

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Club $\mathfrak{S}$	Short Name	Characterises
Identities	**Id**	BI-algebras
Bijections	**Bij**	BCI-algebras
Monotone Injections	**Minj**	BKI-algebras
Injections	**Inj**	BCKI-algebras
Surjections	**Srj**	BCWI-algebras
Functions	**Fun**	BCKWI-algebras

Figure 1: Table of combinatory completeness results.

occurs exactly once in t (see e.g., [6, 3]). As far as we are aware, no satisfying answer to the wider question exists in the literature.

We improve the situation by giving a general notion of combinatory completeness and using it to obtain a number of combinatory completeness results in a systematic fashion. Specifically, we obtain combinatory completeness results characterising applicative systems with B and I combinators together with most subsets of the combinators C,K, and W.

Central to our approach is the notion of faithful cartesian club, which is a sort of well-behaved subcategory of the category **Fun** of functions between sets $\underline{n} = \{1,..,n\}$. Every faithful cartesian club $\mathfrak{S}$ determines a notion of structured multicategory whose instances are called $\mathfrak{S}$-multicategories [5]. We work with a more abstract notion of applicative system, in which the carrier A becomes an object of some ambient $\mathfrak{S}$-multicategory $\mathscr{M}$ and application becomes a morphism $\bullet \in \mathscr{M}(A,A;A)$. We give suitable analogues of the (regular) polynomials, of the notions of computability and combinatory completeness, and of the combinators $\mathsf{B},\mathsf{C},\mathsf{K},\mathsf{W}$ and I in this new setting.

We obtain a number of combinatory completeness results. First, we characterise those applicative systems that are *weakly* $\mathfrak{S}$-combinatory complete for any faithful cartesian club $\mathfrak{S}$, where this means that only the regular $\mathfrak{S}$-polynomials are computable. This is in contrast to $\mathfrak{S}$-combinatory completeness, where all of the $\mathfrak{S}$-polynomials are computable. Our results are summarised in Figure 1. The first column indicates a class of morphisms in **Fun** that forms a faithful cartesian club, which we refer to by the short name given in the second column. The third column tells us which combinators characterise the associated notion of combinatory completeness. For example, the second row states that an applicative system in a **Bij**-multicategory is weakly **Bij**-combinatory complete if and only if it is a BCI-algebra. We show that for any faithful cartesian club $\mathfrak{S}$ that contains the bijections, weak $\mathfrak{S}$-combinatory completeness and $\mathfrak{S}$-combinatory completeness coincide.

If we restrict our attention to the **Fun**-multicategory Set with sets as objects and with functions $f : A_1 \times \cdots \times A_n \to B$ as morphisms $f \in \mathsf{Set}(A_1,..,A_n;B)$ then we recover the classical notion of applicative system and of the $\mathsf{B},\mathsf{C},\mathsf{K},\mathsf{W}$, and I combinators. All of our combinatory completeness results specialise to the classical setting. For example, we obtain Theorem 1 as an instance of the fact that an applicative system in a **Fun**-multicategory is **Fun**-combinatory complete if and only if it is a BCKWI-algebra. In this way, our results apply to the classical notion of applicative system as a set equipped with a binary operation.

A preprint containing a detailed presentation of our results is available [4].

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