Definable Markov Categories

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We construct two Markov categories whose morphisms are definable in o-minimal structures. The probabilistic category $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, over the structure of globally subanalytic sets, has morphisms given by Markov kernels with constructible densities on definable latent spaces; composition corresponds to a fibre-product construction on latent spaces, and the Cluckers–Miller stability theorem ensures that integration against presented kernels preserves the function class. This Markov category does not have conditionals, but is positive and causal, as a subcategory of BorelStoch. The possibilistic category $\mathrm{DefRel}_+(\mathcal{M})$, over an arbitrary o-minimal structure $\mathcal{M}$, has morphisms given by definable total relations; it has conditionals, thus positivity and causality. Both constructions exploit model-theoretic tameness — cell decomposition and closure under integration (probabilistic) or projection (possibilistic) — and we isolate four hypotheses on an abstract function class from which the categorical structure follows.

1 Introduction

Markov categories, introduced by Fritz [Fri20], provide a synthetic framework for probability theory in which the basic operations — composing stochastic processes, copying random variables, and marginalising — are expressed as categorical structure. The key examples are BorelStoch (Markov kernels on standard Borel spaces) and FinStoch (stochastic matrices on finite sets). These categories are large: their morphisms range over all measurable or finite-probability structures, with no tameness constraints.

In this paper we construct two Markov categories whose morphisms are *definable* in an o-minimal structure $\mathcal{M}$ — a setting in which definable sets decompose into finitely many cells and are closed under projections — ensuring that all probability distributions involved have tame, finitely described behaviour. Crucially, associated classes of functions on definable sets are closed under parametric integration, making composition of stochastic kernels well-defined. A key finding is that this tameness comes at a cost: the probabilistic category does not have conditionals (Proposition 3.20), revealing a structural tension between definability and Bayesian inversion.

The probabilistic case (§3). We define $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, a Markov category whose morphisms $X \to Y$ are Markov kernels admitting a *latent-variable presentation*: a finite family of definable latent spaces $U_i \subseteq X \times \mathbb{R}^{m_i}$, nonnegative constructible density functions $g_i \in \mathcal{C}(U_i)$, and definable output maps $h_i : U_i \to Y$, such that the kernel is given by

$$\kappa(x,B) = \sum_i \int_{(U_i)_x} \mathbf{1}_B(h_i(x,u))\, g_i(x,u)\, du.$$

Here the densities g_i are *constructible functions* in the sense of Cluckers–Miller [CM11]: sums of products of globally subanalytic functions and their logarithms. This class is closed under parametric integration [CM11, Thm. 1.3], which is the key property making composition work. We prove that composition corresponds to a fibre-product construction on latent spaces (Theorem 3.8), and that $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is a Markov category (Proposition 3.11).

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The construction is axiomatic: we isolate four hypotheses (H1)–(H4) on an abstract function class $\mathcal{C}_{\mathcal{M}}$ (Definition 3.1) and derive the Markov category structure from these, then verify them for the Cluckers–Miller class. This separates the categorical argument from the model-theoretic input.

The possibilistic case (§4). We construct $\mathrm{DefRel}_+(\mathcal{M})$, a possibilistic Markov category whose morphisms are definable total relations with nonempty fibres. This is a sub-Markov-category of the Kleisli category of the nonempty powerset monad P_+ on Set [Fri20, Cor. 3.2], restricted to definable objects and morphisms. Unlike the probabilistic case, $\mathrm{DefRel}_+(\mathcal{M})$ has conditionals (Proposition 4.4) and hence is positive and causal.

The definable setting. Both constructions rely on the same model-theoretic ingredients: closure of definable sets under projections (for composition) and cell decomposition (for tameness). In the possibilistic case, the fibre construction gives conditionals directly (Proposition 4.4); in the probabilistic case, conditioning requires dividing by a marginal density, which leaves the constructible function class. The parallel is summarised in Remark 4.5. The relationship between the two categories — in particular, whether a support functor $\mathrm{DefStoch}(\mathcal{M}) \to \mathrm{DefRel}_+(\mathcal{M})$ preserves the Markov structure — remains open.

Related work. The connection between o-minimal definability and measure theory is developed by Kaiser [Kai12], who introduces $\mathcal{M}$*-tame* measures and proves that the Lebesgue measure is strongly $\mathbb{R}_{\mathrm{an}}$-tame. Our latent-variable presentation can be seen as a syntactic counterpart to Kaiser's semantic notion; the relationship is discussed in Remark 3.22. Categorical probability in the Markov categories framework is surveyed by Fritz [Fri20], with extensions by Fritz–Rischel [FR20], Moss–Perrone [MP23], and Perrone [Per24]. Quasi-Borel spaces [HKSY17] take a complementary approach: they *enlarge* the category of standard Borel spaces, adding objects such as function spaces to obtain cartesian closure. BorelStoch embeds faithfully into their Markov category.

2 Preliminaries

2.1 O-minimal structures

An *o-minimal structure* on $(\mathbb{R}, <)$ is an expansion $\mathcal{M}$ of the real ordered field such that the subsets of $\mathbb{R}$ definable in $\mathcal{M}$ are exactly the finite unions of points and open intervals. A set $X \subseteq \mathbb{R}^n$ is *definable* (in $\mathcal{M}$) if it can be defined by a first-order formula in the language of $\mathcal{M}$ with parameters from $\mathbb{R}$; maps and families are definable if their graphs are. We refer to [Dri98] for the general theory and recall the main properties used below. O-minimal structures can be seen as a realisation of Grothendieck's programme of *géométrie modérée* (tame topology) [Gro97].

Cell decomposition [Dri98, Ch. 3]: every definable set $X \subseteq \mathbb{R}^n$ admits a finite partition into *cells* — sets defined inductively as points or open intervals in $\mathbb{R}$, and graphs or open bands between continuous definable functions on lower-dimensional cells. Every definable function is piecewise continuous on cells. In particular, definable sets have finitely many connected components. *Definable choice* [Dri98, Ch. 6]: if $S \subseteq X \times Y$ is definable with nonempty fibres S_x for all $x \in \pi_X(S)$, there exists a definable function $f : \pi_X(S) \to Y$ selecting an element from each fibre. Definable sets are closed under Boolean combinations, products, and projections.

The primary structure in this paper is $\mathbb{R}_{\mathrm{an}}$, the expansion of $(\mathbb{R}, <, +, \cdot)$ by all restricted analytic functions $f : [-1,1]^n \to \mathbb{R}$. The sets definable in $\mathbb{R}_{\mathrm{an}}$ are precisely the *globally subanalytic sets*; all definable subsets of $\mathbb{R}^n$ are equipped with their Borel σ-algebra, and globally subanalytic sets are standard Borel spaces. We also use $\mathbb{R}_{\mathrm{an,exp}} = (\mathbb{R}_{\mathrm{an}}, \exp)$, which is o-minimal [DM94] and contains both the globally subanalytic functions and the unrestricted logarithm. In Section 3, $\mathcal{M} = \mathbb{R}_{\mathrm{an}}$; in Section 4, $\mathcal{M}$ is an arbitrary o-minimal structure.

2.2 Markov categories

A *Markov category* [Fri20, Def. 2.1] is a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ in which every object X carries a commutative comonoid structure $\mathrm{copy}_X : X \to X \otimes X$ and $\mathrm{del}_X : X \to I$, compatible with the monoidal product, such that del is natural: $\mathrm{del}_Y \circ f = \mathrm{del}_X$ for every morphism $f : X \to Y$. Intuitively, copy_X duplicates a random variable and del_X discards it; naturality of del expresses normalisation.

A morphism f is *deterministic* if $\mathrm{copy}_Y \circ f = (f \otimes f) \circ \mathrm{copy}_X$ [Fri20, Def. 10.1]. The deterministic morphisms form a cartesian monoidal subcategory.

Example 2.1. BorelStoch, the Kleisli category of the Giry monad [Gir82] on standard Borel spaces, is a Markov category whose morphisms are Markov kernels and whose deterministic morphisms are measurable functions. FinStoch (finite sets, stochastic matrices) is another standard example.

By [Fri20, Cor. 3.2], the Kleisli category of any symmetric monoidal affine monad on a Markov category is again a Markov category. The constructions in §3 and §4 are modelled on this pattern, though neither is literally a Kleisli category (see Remark 4.2).

2.3 Constructible functions

Globally subanalytic functions are not closed under parametric integration: the integral $\int_0^x f(t)\,dt$ of a subanalytic integrand may involve logarithms [LR97]. This motivates the following class.

Definition 2.2 ([CM11, Def. 1.2])**.** For each globally subanalytic set X, the *constructible functions* $\mathcal{C}(X)$ form the $\mathbb{R}$-algebra generated by all globally subanalytic functions on X and the functions $x \mapsto \log f(x)$ for $f : X \to (0, \infty)$ globally subanalytic. Every $g \in \mathcal{C}(X)$ has the form $g(x) = \sum_i f_i(x) \prod_j \log f_{ij}(x)$ with f_i subanalytic and f_{ij} positive subanalytic.

Recall 2.3 ([CM11, Thm. 1.3])**.** *If $g \in \mathcal{C}(X \times \mathbb{R}^m)$ and the integral $\int_{\mathbb{R}^m} g(x,u)\,du$ converges for each x, then $x \mapsto \int_{\mathbb{R}^m} g(x,u)\,du \in \mathcal{C}(X)$.*

This stability under integration is the key closure property: constructible functions are the smallest class of functions that contains all globally subanalytic functions and is closed under parametric integration.

Recall 2.4 (Cell preparation [CM11, Thm. 3.11])**.** *After a definable coordinate change (power substitution), on each cell of a finite subanalytic cell decomposition, every constructible function takes the form $\sum_i u_i(\tilde{x}) \prod_j |\tilde{x}_j|^{\alpha_j} (\log|\tilde{x}_j|)^{\ell_j}$ with $\alpha_j \in \mathbb{Q}$, $\ell_j \in \mathbb{N}_0$, u_i subanalytic units, and $\tilde{x}$ the prepared coordinates.*

The restriction to *rational* exponents is a structural feature: constructible functions exhibit power-law behaviour $|x|^{p/q}$ but never irrational exponents such as $|x|^{\pi}$. This fact is used in §3.6 to exhibit measures outside $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$.

3 Definable stochastic kernels

We now construct $\mathrm{DefStoch}(\mathcal{M})$, a Markov category whose morphisms are stochastic kernels with constructible densities on definable latent spaces. The definition is axiomatic: we state four hypotheses on a function class $\mathcal{C}_{\mathcal{M}}$ and derive the Markov category structure, then instantiate for the Cluckers–Miller constructible functions.

3.1 Admissible function classes

Definition 3.1 (Admissible function class)**.** An *admissible function class* for $\mathcal{M}$ is an assignment $X \mapsto \mathcal{C}_{\mathcal{M}}(X)$ of an $\mathbb{R}$-algebra of Borel measurable functions on X, for each definable set X, satisfying:

(H1) Algebra structure. $\mathcal{C}_{\mathcal{M}}(X)$ is an $\mathbb{R}$-subalgebra of the algebra of all Borel measurable functions $X \to \mathbb{R}$.

(H2) Definable functions. Every function $f : X \to \mathbb{R}$ definable in $\mathcal{M}$ lies in $\mathcal{C}_{\mathcal{M}}(X)$.

(H3) Pullback stability. If $h : X \to Y$ is definable and $g \in \mathcal{C}_{\mathcal{M}}(Y)$, then $g \circ h \in \mathcal{C}_{\mathcal{M}}(X)$.

(H4) Integration closure. If $U \subseteq X \times \mathbb{R}^m$ is definable and $g \in \mathcal{C}_{\mathcal{M}}(U)$ is fibrewise integrable, then $x \mapsto \int_{\mathbb{R}^m} \tilde{g}(x,u)\,du \in \mathcal{C}_{\mathcal{M}}(X)$, where $\tilde{g}$ extends g by zero outside U (note $\tilde{g} = g \cdot \mathbf{1}_U \in \mathcal{C}_{\mathcal{M}}(X \times \mathbb{R}^m)$ when $\mathbf{1}_U$ is in $\mathcal{C}_{\mathcal{M}}$, which holds whenever U is definable, by (H2)).

Remark 3.2. With (H1) requiring all $\mathcal{M}$-definable functions, (H2) is formally contained in (H1). We state it separately because individual results reference different hypotheses: the composition theorem (Theorem 3.8) uses only (H1) and (H3), while stabilisation (Proposition 3.7) additionally requires (H4).

3.2 Presented kernels

Definition 3.3 (Presented kernel)**.** Let X, Y be definable sets. A *presentation* of a kernel $X \to Y$ consists of a finite index set I, integers $m_i \geq 0$, definable sets $U_i \subseteq X \times \mathbb{R}^{m_i}$, nonnegative functions $g_i \in \mathcal{C}_{\mathcal{M}}(U_i)$, and definable maps $h_i : U_i \to Y$, such that for every $x \in X$:

$$\sum_{i \in I} \int_{(U_i)_x} g_i(x,u)\,du = 1, \qquad (U_i)_x := \{u \in \mathbb{R}^{m_i} : (x,u) \in U_i\}. \tag{N}$$

The *induced kernel* is $\kappa(x,B) := \sum_{i \in I} \int_{(U_i)_x} \mathbf{1}_B(h_i(x,u))\, g_i(x,u)\,du$ for Borel $B \subseteq Y$.

A *morphism* $X \to Y$ in $\mathrm{DefStoch}_{\mathcal{C}_{\mathcal{M}}}(\mathcal{M})$ is a Markov kernel admitting such a presentation.

Remark 3.4. Presentations are not unique: a single kernel may admit many presentations with different index sets and latent dimensions. A morphism is the underlying kernel, not the presentation. The latent variables $u \in \mathbb{R}^{m_i}$ are auxiliary random inputs; the density g_i governs their distribution given x, and h_i deterministically produces an output in Y. Singular measures (concentrated on lower-dimensional subsets) require no special treatment: they arise when h_i drops dimension. The map $x \mapsto \kappa(x,B)$ is Borel measurable for each Borel B: it is a finite sum of parametric integrals of measurable functions, hence measurable by Tonelli's theorem.

The following lemma verifies (H3) for the Cluckers–Miller class $\mathcal{C}$. It is immediate from the definition of constructible functions; we state it for reference.

Lemma 3.5 (Pullback stability)**.** *Let $h : X \to Y$ be globally subanalytic and $g \in \mathcal{C}(Y)$ a constructible function. Then $g \circ h \in \mathcal{C}(X)$.*

Proof. By definition, $g(y) = \sum_i f_i(y) \prod_j \log f_{ij}(y)$ with $f_i : Y \to \mathbb{R}$ and $f_{ij} : Y \to (0,\infty)$ globally subanalytic. Pre-composing with h: each $f_i \circ h$ is subanalytic (composition of definable maps), each $f_{ij} \circ h$ is positive subanalytic, and $\log(f_{ij} \circ h) = (\log f_{ij}) \circ h$. So $g \circ h = \sum_i (f_i \circ h) \prod_j \log(f_{ij} \circ h) \in \mathcal{C}(X)$. $\square$

3.3 Composition

Lemma 3.6 (Integration against a presented kernel). *Let $\kappa : X \to Y$ be presented as $(U_i, g_i, h_i)_{i\in I}$. For any nonnegative Borel measurable $\varphi : Y \to [0,\infty]$:*

$$\int_Y \varphi(y)\,\kappa(x,dy) = \sum_{i\in I} \int_{(U_i)_x} \varphi(h_i(x,u))\,g_i(x,u)\,du. \tag{IL}$$

Proof. For $\varphi = \mathbf{1}_B$ this is the definition. Extend to simple functions by linearity and to general nonnegative measurable functions by monotone convergence. By splitting $\varphi = \varphi^+ - \varphi^-$, the identity extends to any κ_x-integrable φ. □

Proposition 3.7 (Stabilisation). *If $\kappa : X \to Y$ is a presented kernel and $\varphi \in \mathcal{C}_{\mathcal{M}}(Y)$ satisfies $\int_Y |\varphi|\,d\kappa_x < \infty$ for all x, then $x \mapsto \int_Y \varphi\,d\kappa_x \in \mathcal{C}_{\mathcal{M}}(X)$.*

Proof. By (IL), $\int_Y \varphi\,d\kappa_x = \sum_i \int_{(U_i)_x} \varphi(h_i(x,u))\,g_i(x,u)\,du$. Each summand lies in $\mathcal{C}_{\mathcal{M}}(X)$: the pullback $\varphi\circ h_i$ is in $\mathcal{C}_{\mathcal{M}}(U_i)$ by (H3), the product $(\varphi\circ h_i)\cdot g_i \in \mathcal{C}_{\mathcal{M}}(U_i)$ by (H1), and $|\varphi\circ h_i|\cdot g_i$ integrates to at most $\int_Y |\varphi|\,d\kappa_x < \infty$, so the parametric integral lies in $\mathcal{C}_{\mathcal{M}}(X)$ by (H4). The finite sum is in $\mathcal{C}_{\mathcal{M}}(X)$ by (H1). □

Theorem 3.8 (Composition closure). *Let $\kappa : X \to Y$ have presentation $(U_i, g_i, h_i)_{i\in I}$ with latent dimensions m_i, and $\lambda : Y \to Z$ have presentation $(V_j, k_j, p_j)_{j\in J}$ with latent dimensions n_j. The Kleisli composite $\lambda\circ\kappa$ has presentation $(W_{ij}, G_{ij}, H_{ij})_{(i,j)\in I\times J}$ where:*

$$W_{ij} := \{(x,u,v) : (x,u) \in U_i,\ (h_i(x,u), v) \in V_j\} \subseteq X \times \mathbb{R}^{m_i+n_j}, \tag{C1}$$

$$G_{ij}(x,u,v) := g_i(x,u)\cdot k_j(h_i(x,u), v), \tag{C2}$$

$$H_{ij}(x,u,v) := p_j(h_i(x,u), v). \tag{C3}$$

The hypotheses used are (H1) *and* (H3) *for constructibility of G_{ij}; closure under composition follows from the explicit fibre-product construction without invoking* (H4).

Proof. We verify that (W_{ij}, G_{ij}, H_{ij}) is a valid presentation inducing $(\lambda\circ\kappa)(x,C) = \int_Y \lambda(y,C)\,\kappa(x,dy)$.

Definability and constructibility. W_{ij} is definable (intersection of preimages under definable maps). $G_{ij} = (g_i\circ\pi)\cdot(k_j\circ\psi) \in \mathcal{C}_{\mathcal{M}}(W_{ij})$ where $\pi(x,u,v) = (x,u)$ and $\psi(x,u,v) = (h_i(x,u), v)$, using (H3) for pullbacks and (H1) for the product. $H_{ij} = p_j\circ\psi$ is definable.

Induced kernel. By (IL) applied to κ with $\varphi(y) = \lambda(y,C)$:

$$(\lambda\circ\kappa)(x,C) = \sum_i \int_{(U_i)_x} \lambda(h_i(x,u), C)\,g_i(x,u)\,du.$$

Substituting the presentation of λ and applying Tonelli (the integrand is nonnegative, index sets finite):

$$= \sum_{(i,j)} \int_{\mathbb{R}^{m_i+n_j}} \mathbf{1}_{(U_i)_x}(u)\,\mathbf{1}_{(V_j)_{h_i(x,u)}}(v)\,\mathbf{1}_C(p_j(h_i(x,u),v))\,k_j(h_i(x,u),v)\,g_i(x,u)\,d(u,v)$$

$$= \sum_{(i,j)} \int_{(W_{ij})_x} \mathbf{1}_C(H_{ij}(x,u,v))\,G_{ij}(x,u,v)\,d(u,v).$$

Normalization. Setting $C = Z$: $\sum_{(i,j)} \int_{(W_{ij})_x} G_{ij} = (\lambda\circ\kappa)(x,Z) = 1$. Composition is well-defined on kernels (the Kleisli integral depends only on κ and λ, not on their presentations), and associativity follows from Fubini. □

3.4 Markov category structure

Every definable map $f : X \to Y$ defines a morphism via the Dirac kernel $\delta_f(x,B) = \mathbf{1}_B(f(x))$, presented with $|I| = 1$, $m_1 = 0$, $g_1 \equiv 1$, $h_1 = f$. In particular, $\mathrm{Def}(\mathcal{M})$ embeds as the deterministic sub-Markov-category.

Proposition 3.9 (Tensor product)**.** *Let $\kappa : X_1 \to Y_1$ and $\lambda : X_2 \to Y_2$ be presented kernels. The product measure $(\kappa \otimes \lambda)((x_1,x_2),-) := \kappa(x_1,-) \otimes \lambda(x_2,-)$ is a presented kernel $X_1 \times X_2 \to Y_1 \times Y_2$, with presentation obtained by taking products of latent spaces and multiplying densities.*

Proof. With presentations $(U_i, g_i, h_i)_{i \in I}$ and $(V_j, k_j, p_j)_{j \in J}$, set

$$W_{ij} := \{((x_1,x_2),u,v) : (x_1,u) \in U_i, (x_2,v) \in V_j\}, \quad G_{ij} := g_i \cdot k_j, \quad H_{ij} := (h_i, p_j).$$

W_{ij} is definable, $G_{ij} \in \mathcal{C}_{\mathcal{M}}(W_{ij})$ by (H3) (pullbacks to the product) and (H1) (product), and H_{ij} is definable. Normalization holds since the total mass factors: $\sum_{ij} \int G_{ij} = (\sum_i \int g_i)(\sum_j \int k_j) = 1 \cdot 1 = 1$. On rectangles $B_1 \times B_2$:

$$(\kappa \otimes \lambda)((x_1,x_2), B_1 \times B_2) = \kappa(x_1,B_1) \cdot \lambda(x_2,B_2),$$

which matches the presentation by Fubini. Agreement on rectangles extends to all Borel sets by the π–λ theorem. □

Lemma 3.10 (Interchange law)**.** *For morphisms $f_1 : X_1 \to Y_1$, $f_2 : Y_1 \to Z_1$, $g_1 : X_2 \to Y_2$, $g_2 : Y_2 \to Z_2$:*

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1).$$

Proof. Both sides are kernels $X_1 \times X_2 \to Z_1 \times Z_2$. On rectangles $C_1 \times C_2$, the LHS equals

$$\int_{Y_1 \times Y_2} f_2(y_1, C_1)\, g_2(y_2, C_2)\, d(f_1(x_1,-) \otimes g_1(x_2,-)),$$

which factors by Fubini into $(f_2 \circ f_1)(x_1, C_1) \cdot (g_2 \circ g_1)(x_2, C_2)$, matching the RHS. Extend by π–λ. □

Proposition 3.11 (Markov category)**.** $\mathrm{DefStoch}_{\mathcal{C}_{\mathcal{M}}}(\mathcal{M})$ *is a Markov category in the sense of [Fri20], with monoidal product $X \otimes Y := X \times Y$, unit $I := \{*\}$,* $\mathrm{copy}_X(x,B) = \mathbf{1}_B(x,x)$, $\mathrm{del}_X(x,\{*\}) = 1$, *and symmetry $\sigma_{X,Y}((x,y),B) = \mathbf{1}_B(y,x)$.*

Proof. All structure morphisms are Dirac kernels of definable maps. The symmetric monoidal structure is inherited from $\mathrm{Def}(\mathcal{M})$. We verify the axioms of [Fri20, Def. 2.1]:

(2.2)–(2.3): Commutative comonoid. Coassociativity, counitality, and commutativity hold since copy_X and del_X are Dirac kernels of the diagonal $x \mapsto (x,x)$ and the unique map $x \mapsto *$.

(2.4): Monoidal compatibility. $\mathrm{del}_{X \times Y}$ sends $(x,y) \mapsto \delta_*$, agreeing with $\mathrm{del}_X \otimes \mathrm{del}_Y$. $\mathrm{copy}_{X \times Y}$ sends $(x,y) \mapsto \delta_{(x,y,x,y)}$; the composite $(\mathrm{id}_X \otimes \sigma_{X,Y} \otimes \mathrm{id}_Y) \circ (\mathrm{copy}_X \otimes \mathrm{copy}_Y)$ sends $(x,y) \mapsto (x,x,y,y) \mapsto (x,y,x,y)$.

(2.5): Naturality of del. For any morphism f, $\mathrm{del}_Y \circ f = \mathrm{del}_X$ holds by the normalisation condition (N).

$\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \times Y}$ (both are the Dirac kernel of the identity). Bifunctoriality is Lemma 3.10. Associativity and unitality of composition hold at the level of Markov kernels by Fubini; Theorem 3.8 ensures that the composites admit presentations, so these operations are well-defined in $\mathrm{DefStoch}_{\mathcal{C}_{\mathcal{M}}}(\mathcal{M})$. □

3.5 Instantiation and examples

Proposition 3.12. *The Cluckers–Miller constructible functions [CM11] form an admissible function class for* $\mathbb{R}_{\text{an}}$*:* (H1) *since* $\mathcal{C}(X)$ *is an* $\mathbb{R}$*-algebra containing all globally subanalytic (=* $\mathbb{R}_{\text{an}}$*-definable) functions by definition,* (H3) *by Lemma 3.5,* (H4) *by [CM11, Thm. 1.3].*

Definition 3.13. $\mathrm{DefStoch}(\mathbb{R}_{\text{an}})$ denotes $\mathrm{DefStoch}_{\mathcal{C}}(\mathbb{R}_{\text{an}})$ with $\mathcal{C}$ the Cluckers–Miller constructible functions. More generally, $\mathrm{DefStoch}(\mathcal{M})$ denotes $\mathrm{DefStoch}_{\mathcal{C}}(\mathcal{M})$ for any o-minimal $\mathcal{M}$ equipped with an admissible function class.

For strict expansions of $\mathbb{R}_{\text{an}}$ (e.g., $\mathbb{R}_{\text{an,exp}}$), the Cluckers–Miller class $\mathcal{C}$ need not be admissible: (H1) may fail since the exponential function is $\mathbb{R}_{\text{an,exp}}$-definable but not constructible, and (H3) may fail for pullbacks along non-subanalytic definable maps. Extending Definition 3.1 to richer structures requires a correspondingly richer function class.

Remark 3.14 (Initiality)**.** For $\mathbb{R}_{\text{an}}$, the Cluckers–Miller class is the *smallest* admissible function class: it is the smallest $\mathbb{R}$-algebra containing all globally subanalytic functions and closed under parametric integration [CM11, p. 3]. Any admissible class $\mathcal{C}'$ contains the subanalytic functions by (H1) and is closed under parametric integration by (H4), so $\mathcal{C} \subseteq \mathcal{C}'$, hence $\mathrm{DefStoch}_{\mathcal{C}}(\mathbb{R}_{\text{an}}) \subseteq \mathrm{DefStoch}_{\mathcal{C}'}(\mathbb{R}_{\text{an}})$. Thus $\mathrm{DefStoch}(\mathbb{R}_{\text{an}})$ is the smallest sub-Markov-category of BorelStoch obtainable from $\mathrm{Def}(\mathbb{R}_{\text{an}})$ by the axiomatic framework.

Proposition 3.15 (Kernel types)**.** *The following kernel types are morphisms in* $\mathrm{DefStoch}_{\mathcal{C}_{\mathcal{M}}}(\mathcal{M})$*:*

1. ***Deterministic**: $\delta_f(x,B) = \mathbf{1}_B(f(x))$ for definable $f : X \to Y$. Presented with $|I| = 1$, $m_1 = 0$, $g_1 \equiv 1$, $h_1 = f$.*
2. ***Absolutely continuous**: $\kappa(x,B) = \int_B g(x,y)\,dy$ for $g \in \mathcal{C}_{\mathcal{M}}(X \times Y)$ with $g \geq 0$ and $\int_Y g(x,y)\,dy = 1$, where $Y \subseteq \mathbb{R}^k$ has dimension k. Presented with $U = X \times Y$, $h = \pi_Y$.*
3. ***Singular**: $\kappa(x,-)$ is the pushforward of a constructible density on a definable set $U_x \subseteq \mathbb{R}^m$ through a definable map $h : U \to Y$ with $\dim h(U_x) < \dim Y$. This is a single-component presentation.*
4. ***Finite mixture**: concatenation of presentations with constructible weight functions summing to* 1. *(Given κ_k with weights $w_k \in \mathcal{C}_{\mathcal{M}}(X)$, $w_k \geq 0$, $\sum_k w_k = 1$, the mixture has presentation $(w_k \cdot g_{k,i})$ on the disjoint union of latent spaces.)*

Crucially, type labels are *not* preserved under composition:

Example 3.16 (Dimension jumping)**.** Let $X = (0,1)$, $Y = Z = \mathbb{R}^2$. Define singular kernels $\kappa_1 : X \to Y$ by $\kappa_1(x,-) = \mathrm{Uniform}(\{(t,0) : t \in [0,x]\})$ and $\kappa_2 : Y \to Z$ by $\kappa_2(y,-) = \mathrm{Uniform}(\{(y_1,s) : s \in [0,1]\})$ (ignoring y_2). Both have 1-dimensional support in $\mathbb{R}^2$.

By Theorem 3.8, the composite has latent space $W = \{(x,t,s) : 0 \leq t \leq x, 0 \leq s \leq 1\}$ with density $G(x,t,s) = 1/x$ and output map $H(x,t,s) = (t,s)$. The induced kernel is $(\kappa_2 \circ \kappa_1)(x,-) = \mathrm{Uniform}([0,x] \times [0,1])$, with density

$$f(x; z_1, z_2) = \frac{1}{x} \cdot \mathbf{1}_{[0,x]}(z_1) \cdot \mathbf{1}_{[0,1]}(z_2),$$

which is absolutely continuous (2-dimensional support) and constructible (subanalytic). In particular, type labels (singular, absolutely continuous) are properties of presentations, not morphism invariants.

Example 3.17 (Running example)**.** The kernel $k : (0,1] \to [0,1]$ given by $k(x,-) = \mathrm{Uniform}([0,x])$ has presentation $g(x,t) = 1/x$ on $U = \{0 \leq t \leq x\}$ with $h(x,t) = t$. By induction, its n-fold composition has density

$$k^{\circ n}(x,t) = \frac{(\log(x/t))^{n-1}}{(n-1)!\,x} \cdot \mathbf{1}_{(0,x]}(t).$$

The base kernel is subanalytic; the first composition introduces log, illustrating the Lion–Rolin phenomenon [LR97]. All iterates remain constructible, with log-degree growing by 1 per composition.

3.6 Properties and comparison

Every presented kernel is a Markov kernel on standard Borel spaces, giving a faithful identity-on-objects functor $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}}) \to \mathrm{BorelStoch}$ that preserves the Markov category structure. Thus $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is a sub-Markov-category of BorelStoch, selecting those kernels admitting a constructible latent-variable presentation.

Remark 3.18 (Deterministic morphisms)**.** A morphism $\kappa : X \to Y$ is deterministic (in the sense of [Fri20, Def. 10.1]) if and only if it is a Dirac kernel δ_f for some definable $f : X \to Y$. Indeed, determinism forces each κ_x to be a point mass $\delta_{f(x)}$. For definability: given a presentation (U_i, g_i, h_i), on any cell of a cell decomposition of U_i where $g_i \not\equiv 0$, the output map $h_i(x,u)$ must equal $f(x)$ for all u in the cell fibre (since $g_i \geq 0$ is continuous on cells). Definable choice gives a section $u_0 : X \to U_i$ with $f(x) = h_i(x, u_0(x))$, which is definable.

Remark 3.19. $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is *positive* [Fri20, Def. 11.22] and *causal* [Fri20, Def. 11.31]: positivity is inherited via the faithful Markov embedding $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}}) \hookrightarrow \mathrm{BorelStoch}$ [Fri20, Rem. 11.26], and causality likewise transfers along faithful Markov functors (it is a universal Horn condition); BorelStoch is causal by [Fri20, Prop. 11.34] since it has conditionals. Thus $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is positive and causal but does not have conditionals (Proposition 3.20): the conditionals exist in BorelStoch (regular conditional distributions) but need not restrict to $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$: there are presented joints whose regular conditionals are not morphisms in $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ (Proposition 3.20).

We now turn to the main negative result of this section.

Proposition 3.20 (No conditionals)**.** $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ *does not have conditionals in the sense of [Fri20, Def. 11.5] (see also [CJ19] for the connection between conditionals and Bayesian inversion).*

Proof sketch. Consider the joint $f : \{*\} \to (0,1) \times (0,1)$ with density $\frac{4}{3}(-\log y) \cdot \mathbf{1}_{0<y<x<1}$; this is a presented kernel (the density is constructible and the domain is definable). The X-marginal $f_X(dx) = \frac{4}{3} x(1 - \log x)\, dx$ has full support on $(0,1)$. If any presented kernel κ satisfied the conditional equation [Fri20, Def. 11.5], it would agree with the measure-theoretic conditional for a.e. x [Fri20, Prop. 13.7]. Stabilisation (Proposition 3.7) with $\varphi(y) = y$ would then give a constructible function $F(x) = \int y\, \kappa(x, dy)$ equal a.e. to

$$\frac{x(1 - 2\log x)}{4(1 - \log x)}.$$

Since constructible functions are continuous on cells, F equals this expression on some interval $(0, \delta)$. But $F(x) - x/2 = -x/\big(4(1 - \log x)\big) \sim -x \cdot (-\log x)^{-1}/4$ as $x \to 0^+$. Cell preparation (Recall 2.4) requires all log-exponents $\ell_j \geq 0$, so a term $(-\log x)^{-1}$ cannot appear in any cell preparation of a constructible function. Thus F is not constructible — contradiction. □

Lemma 3.21 (Density lemma)**.** *If μ is a presented probability measure on $\mathbb{R}^n$ that is absolutely continuous with respect to Lebesgue measure, then its Radon–Nikodym density $d\mu/d\mathrm{Leb}$ lies in $\mathcal{C}_{\mathcal{M}}(\mathbb{R}^n)$.*

Proof sketch. Decompose the presentation by rank of h_i on cells. Rank $< n$ components are singular (zero contribution). Equal-dimensional components use the change-of-variables formula with definable inverse. Submersive components use the implicit function theorem and (H4). □

The density lemma is also used in the conclusion to rule out conditional products: if the conditional-independence joint were presented, its density would be constructible, but cell preparation forbids the required terms.

Remark 3.22 (Tame measures)**.** Kaiser [Kai12] introduces $\mathcal{M}$-*tame* measures: a Borel measure μ on a definable set X is $\mathcal{M}$-tame if for every $\mathcal{M}$-definable family $\{f_t\}_{t\in T}$ of integrable functions, the parametric integral $t \mapsto \int_X f_t\, d\mu$ is definable in some o-minimal expansion $\mathcal{M}^*$ of $\mathcal{M}$. Every presented kernel κ_x in $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is $\mathbb{R}_{\mathrm{an}}$-tame: integrating an $\mathbb{R}_{\mathrm{an}}$-definable family against κ_x yields a constructible function by Proposition 3.7, and constructible functions are $\mathbb{R}_{\mathrm{an,exp}}$-definable.

Whether the converse holds — whether every $\mathbb{R}_{\mathrm{an}}$-tame probability measure on a definable set admits a presentation — is open. The measure μ with density $(\pi+1)x^{\pi}$ on $(0,1)$ witnesses that the inclusion is strict. It is not presentable: if it were, the CDF $t^{\pi+1}$ would be constructible by (H4), but cell preparation forces rational exponents, giving $\pi \notin \mathbb{Q}$ a contradiction. However, μ *is* $\mathbb{R}_{\mathrm{an}}$-tame with $\mathcal{M}^* = \mathbb{R}_{\mathrm{an,exp}}$: on each cell of the Lion–Rolin preparation [LR97], the factor x^{π} is absorbed into restricted analytic terms — via the binomial expansion $(1+z)^{\pi}$ for $|z| \le \frac{1}{2}$ near nonzero centres, and the irrationality of π to avoid resonances near zero — reducing to $\mathbb{R}_{\mathrm{an}}$-definable integrands whose parametric integrals are $\mathbb{R}_{\mathrm{an,exp}}$-definable.

4 The possibilistic case

We now construct $\mathrm{DefRel}_+(\mathcal{M})$, a possibilistic analogue of $\mathrm{DefStoch}(\mathcal{M})$ in which probability measures are replaced by nonempty definable subsets.

4.1 Definition

Definition 4.1. $\mathrm{DefRel}_+(\mathcal{M})$ is the category whose objects are nonempty definable sets in $\mathcal{M}$ and whose morphisms $X \to Y$ are definable relations $R \subseteq X \times Y$ with nonempty fibres: $R_x = \{y \in Y : (x,y) \in R\} \neq \varnothing$ for all $x \in X$. Composition is relational:

$$(S \circ R)_x = \bigcup_{y \in R_x} S_y = \pi_Z\{z : \exists y,\ (x,y) \in R \wedge (y,z) \in S\}.$$

The identity on X is the diagonal Δ_X.

Composition is well-defined: the set $R \times_Y S = \{(x,y,z) : (x,y) \in R, (y,z) \in S\}$ is definable, and its projection $\pi_{X\times Z}(R \times_Y S)$ is definable by o-minimality. This is the possibilistic analogue of the integration closure (H4) that makes $\mathrm{DefStoch}(\mathcal{M})$ work: projection closure plays the role of integration closure.

Remark 4.2. The nonempty powerset monad P_+ on Set sends $X \mapsto 2^X \setminus \{\varnothing\}$. Its Kleisli category $\mathrm{Kl}(P_+)$ has morphisms $X \to Y$ given by functions $X \to P_+(Y)$, i.e., total relations with nonempty fibres. $\mathrm{DefRel}_+(\mathcal{M})$ is the subcategory of $\mathrm{Kl}(P_+)$ on definable objects, with morphisms restricted to definable relations. Since $P_+(X)$ is not a definable set for infinite X, the monad P_+ does not restrict to $\mathrm{Def}(\mathcal{M})$; the category $\mathrm{DefRel}_+(\mathcal{M})$ is constructed directly rather than as a Kleisli category.

4.2 Markov category structure

Proposition 4.3. $\mathrm{DefRel}_+(\mathcal{M})$ *is a Markov category with monoidal product* $X \otimes Y := X \times Y$*, unit* $I := \{*\}$*,* $\mathrm{copy}_X(x) = \{(x,x)\}$*,* $\mathrm{del}_X(x) = \{*\}$*, and symmetry* $\sigma_{X,Y}(x,y) = \{(y,x)\}$*.*

Proof. All structure morphisms are deterministic (singleton-fibre) and definable. The monoidal product of morphisms is $(R \otimes S)_{(x_1,x_2)} = R_{x_1} \times S_{x_2}$, which is definable with nonempty fibres. The axioms of [Fri20, Def. 2.1] are verified as in Proposition 3.11: comonoid laws hold since copy and del are given by the diagonal and terminal maps, monoidal compatibility is direct, and naturality of del holds since every fibre is nonempty ($\mathrm{del}_Y(R_x) = \{*\} = \mathrm{del}_X(x)$). □

A morphism $R : X \to Y$ is *deterministic* iff $|R_x| = 1$ for all x, i.e., R is the graph of a definable function. The deterministic sub-Markov-category is $\mathrm{Def}(\mathcal{M})$.

$\mathrm{DefRel}_+(\mathcal{M})$ is *not* cartesian: a morphism $I \to X \times Y$ is a nonempty definable subset $S \subseteq X \times Y$, which need not be a product. For instance, the simplex $\{(x,y) \in [0,1]^2 : x + y \leq 1\}$ expresses correlated nondeterminism.

Proposition 4.4. $\mathrm{DefRel}_+(\mathcal{M})$ *has conditionals, and hence is positive and causal [Fri20, Lem. 11.24, Prop. 11.34].*

Proof. Given $f : A \to X \times Y$ in $\mathrm{DefRel}_+(\mathcal{M})$, define $f|_X : X \times A \to Y$ by

$$f|_X(x,a) = \begin{cases} \{y : (x,y) \in f(a)\} & \text{if } x \in \pi_X(f(a)), \\ Y & \text{otherwise.} \end{cases}$$

When $x \in \pi_X(f(a))$, the fibre is nonempty by definition of projection. When $x \notin \pi_X(f(a))$, we use Y (nonempty by assumption on objects). Both cases are definable, so $f|_X$ is a morphism in $\mathrm{DefRel}_+(\mathcal{M})$. For the chain rule, note that composing with the X-marginal $f_X(a) = \pi_X(f(a))$ restricts to $x \in f_X(a)$, so the default case is never reached: $\{(x,y) : x \in f_X(a),\ y \in f|_X(x,a)\} = f(a)$. □

4.3 Definable-specific features

Beyond the generic Markov category structure (which $\mathrm{Kl}(P_+)$ on Set already has), the definable setting provides:

Cell decomposition bounds complexity. Every morphism $R : X \to Y$ decomposes into finitely many cells by [Dri98, Ch. 3]. The fibres R_x have finitely many connected components and bounded dimension. The function $x \mapsto \dim(R_x)$ takes finitely many values, and its level sets form a definable partition of X — a possibilistic analogue of the kernel-type classification (Proposition 3.15).

Definable choice. By [Dri98, Ch. 6], every morphism $R : X \to Y$ admits a deterministic section: a definable function $g : X \to Y$ with $g(x) \in R_x$ for all x. This property is specific to o-minimal structures and fails in general topological or measurable settings.

Remark 4.5. Let us compare the two Markov categories constructed in this paper:

	$\mathrm{DefStoch}(\mathcal{M})$	$\mathrm{DefRel}_+(\mathcal{M})$
Morphisms	presented kernels	definable total relations
Composition	fibre product of latent spaces	relational (projection)
Closure mechanism	integration closure (H4)	projection (o-minimality)
Conditionals	no (Proposition 3.20)	fibre construction
Positive, causal	yes (Remark 3.19)	yes (via conditionals)
Randomness pushback	open (§5)	yes (definable choice)
Deterministic	definable functions	definable functions

The contrast between the two rows “Conditionals” and “Closure mechanism” is not coincidental: possibilistic conditioning is a set-theoretic operation (taking fibres of a relation, which preserves definability), while probabilistic conditioning requires division by a marginal density, which can leave the constructible function class.

The *support map* $\mathrm{supp} : \mathrm{DefStoch}(\mathcal{M}) \to \mathrm{DefRel}_+(\mathcal{M})$, sending each kernel to the definable relation $\mathrm{supp}(\kappa) = \{(x,y) : y \in \mathrm{supp}(\kappa_x)\}$, preserves identities, copy, delete, tensor, and symmetry (all structure morphisms are Dirac, whose support is a function graph). For composition, the inclusion $\mathrm{supp}(\lambda \circ \kappa) \subseteq \overline{\mathrm{supp}(\lambda) \circ \mathrm{supp}(\kappa)}$ holds (fibrewise closure in the codomain), but the closure cannot be removed: a kernel $\kappa = \mathrm{Uniform}(0,1)$ composed with $\lambda_y = \delta_{1/y}$ into $[1,\infty)$ has $\mathrm{supp}(\lambda \circ \kappa) = [1,\infty)$ while $\mathrm{supp}(\lambda) \circ \mathrm{supp}(\kappa) = (1,\infty)$. Conversely, equality can fail even with closure: a measure-zero set in $\mathrm{supp}(\kappa_x)$ can affect the relational composite without affecting the probabilistic one.

5 Conclusion

We have constructed two Markov categories of definable morphisms: $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, whose morphisms are stochastic kernels with constructible latent-variable presentations, and $\mathrm{DefRel}_+(\mathcal{M})$, whose morphisms are definable total relations. Both exploit the same model-theoretic ingredients — projection closure and cell decomposition — and the axiomatic framework (H1)–(H4) isolates what is needed from the function class.

The two categories exhibit an asymmetry in their Markov-theoretic properties. The possibilistic category $\mathrm{DefRel}_+(\mathcal{M})$, a sub-Markov-category of $\mathrm{Kl}(P_+)$ on Set restricted to definable objects, has conditionals, positivity, and causality (Proposition 4.4). The probabilistic category $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, by contrast, does *not* have conditionals (Proposition 3.20): the obstruction is that conditioning introduces reciprocals of constructible functions, and the cell preparation theorem forces all log-exponents to be nonnegative integers, ruling out terms of the form $(-\log x)^{-1}$. This is a *definability* obstruction, not a measure-theoretic one — the conditionals exist in BorelStoch but do not restrict to $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$. The same obstruction rules out *conditional products* in the sense of [Fri20, Prop. 12.9]: a pair of presented kernels with common marginal $r(w) = \frac{4}{3}w(1-\log w)$ has a conditional-independence joint that is absolutely continuous on $(0,1)^3$ (since both conditionals have densities on open subsets of $(0,1)$) with density involving $1/(w(1-\log w))$; the density lemma (Lemma 3.21) forces this to be constructible if the joint is presented, but cell preparation forbids the term $(-\log w)^{-1}$ (by an analogous cell preparation argument). Thus even the strictly weaker notion of conditional products for specific well-behaved pairs fails.

The dilations framework of Fritz–Gonda et al. [FGHL+23] gives a reformulation. Positivity is equivalent to the existence of initial dilations for all *deterministic* morphisms [FGHL+23, Prop. 4.12]. Conditionals guarantee initial dilations for *all* morphisms [FGHL+23, Prop. 4.13]. Since $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is positive but lacks conditionals, it has initial dilations for deterministic morphisms (definable maps). For non-deterministic kernels, the situation is open: the standard construction of initial dilations via the input-output copy [FGHL+23, Prop. 4.13] requires the conditional, which would introduce the forbidden $(-\log x)^{-1}$ terms, but an alternative construction not passing through conditionals has not been ruled out.

This failure mode is distinct from that of quasi-Borel spaces [HKSY17]: in QBStoch, the privacy equation forces non-positivity [SSSW21, Props. 5.2, 5.8], so the obstruction is global (positivity itself fails). In $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, the failure is localised: conditionals fail already at pairs of intervals, but positivity and causality are fully preserved. Among the Markov categories in which conditionals are known to fail, $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ retains the most structure: deterministic marginal independence, parametrised equality strengthening, and positivity all survive.

Despite this, the obstruction is specific to the *conditional independence* condition, not to marginal matching per se. For the same counterexample pair, the *comonotone coupling* — the presented kernel on latent space $U = \{(x,w) : 0 < x < w < 1\}$ with density $g(x,w) = \frac{4}{3}(-\log x)$ and output map $h(x,w) = (x,w,x)$ — is a valid morphism $I \to X \otimes W \otimes Y$ in $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ whose $X \otimes W$ and $W \otimes Y$ marginals match those of the counterexample pair. This coupling sets $Y = X$ almost surely (maximal dependence, violating conditional independence) and is singular with respect to Lebesgue measure on $(0,1)^3$, so the density lemma does not apply. The existence of such definable couplings, in the absence of conditional products, suggests that the space of definable couplings for a given marginal pair may carry useful categorical structure, even when the unique conditional-independence coupling is not presentable.

We close with four open directions.

Support map. The map $\mathrm{supp} : \mathrm{DefStoch}(\mathcal{M}) \to \mathrm{DefRel}_+(\mathcal{M})$ preserves all Markov category structure except composition, where fibrewise closure is needed and even then the inclusion can be strict (Remark 4.5). Whether supp can be made a strict or lax Markov functor via a modified relational composition is open.

Tame measures and the presentation gap. Every presented kernel in $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is $\mathbb{R}_{\mathrm{an}}$-tame in Kaiser's sense (Remark 3.22), but the converse fails: the measure with density $(\pi+1)x^{\pi}$ on $(0,1)$ is $\mathbb{R}_{\mathrm{an}}$-tame (with uniform witness $\mathbb{R}_{\mathrm{an,exp}}$) yet not presentable (Remark 3.22). This strict gap between semantic tameness and syntactic presentability is open at both ends: whether every tame measure is a limit of presentable ones, and whether the gap persists for absolutely continuous measures with constructible support, remain to be understood.

Randomness pushback. Fritz [Fri20, Def. 11.19] asks whether every morphism factors through a deterministic map and a randomness source. For $\mathrm{DefRel}_+(\mathcal{M})$ the answer is yes, using definable choice for sections. For $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, the question reduces to a transport problem: same-dimensional subanalytic maps can only produce subanalytic densities, but higher-dimensional maps can produce constructible densities (e.g., $T(x,y) = xy$ pushes Leb^2 to $-\log t$). Monomial densities from cell preparation are achievable via products of uniforms; the remaining gap is whether cross-coordinate subanalytic unit factors also admit subanalytic transports.

Beyond $\mathbb{R}_{\mathrm{an}}$. The axiomatic framework applies to any function class satisfying (H1)–(H4), but we have only verified the hypotheses for Cluckers–Miller constructible functions over $\mathbb{R}_{\mathrm{an}}$, where the constructible class is the *smallest* admissible class (Remark 3.14). Other candidates include log-analytic functions in the sense of Lion–Rolin [LR97], which form a strictly larger algebra than the constructible class and would give a larger category but weaker closure properties, and function classes over polynomially bounded o-minimal structures. The conditionals obstruction (Proposition 3.20) is specific to the Cluckers–Miller class and its nonnegative log-exponent constraint; a function class admitting $(-\log x)^{-1}$ might support conditionals, at the cost of losing the clean cell preparation theorem.

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A Proof of Proposition 3.20

We show that a specific morphism in $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ admits no presented kernel satisfying Fritz's conditional equation.

The counterexample. Define $f : \{*\} \to (0,1)\times(0,1)$ by the probability measure with density

$$g(x,y) \;=\; \tfrac{4}{3}(-\log y)\cdot \mathbf{1}_{0<y<x<1}$$

with respect to Lebesgue measure on $(0,1)^2$. This is a presented kernel: $|I| = 1$, latent dimension $m = 2$, domain $U = \{(x,y)\in\mathbb{R}^2 : 0<y<x<1\}$ (definable), density $\frac{4}{3}\log(1/y)$ (constructible, since $1/y$ is positive subanalytic), output map $h(x,y) = (x,y)$ (definable), and $g \geq 0$ on U.

Normalization. $\int_0^x(-\log y)\,dy = x(1-\log x)$ by integration by parts (using $\lim_{y\to 0^+} y\log y = 0$), and $\int_0^1 x(1-\log x)\,dx = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$, giving $\frac{4}{3}\cdot\frac{3}{4} = 1$.

The conditional. The X-marginal is $f_X(dx) = \frac{4}{3}x(1-\log x)\,dx$, which is strictly positive on $(0,1)$. By the disintegration theorem, the conditional $f|_X(x,dy) = \frac{(-\log y)}{x(1-\log x)}\cdot\mathbf{1}_{(0,x)}(y)\,dy$ is uniquely determined for f_X-a.e. x.

Stabilisation test. Suppose for contradiction that some presented kernel κ satisfies Fritz's conditional equation for f. The conditional equation determines $\kappa(x,-)$ for f_X-a.e. x [Fri20, Prop. 13.7], hence for Lebesgue-a.e. x (since f_X is equivalent to Lebesgue). Taking $\varphi(y) = y \in \mathcal{C}_{\mathcal{M}}\big((0,1)\big)$, Proposition 3.7 gives $\Phi(x) := \int y\,\kappa(x,dy) \in \mathcal{C}_{\mathcal{M}}\big((0,1)\big)$. Since Φ is constructible, it is continuous on each cell of a finite cell decomposition; since Φ equals the continuous function $F(x) := \int y\, f|_X(x,dy)$ for a.e. x, we have $\Phi = F$ on $(0,\delta)$ for some $\delta > 0$ (two continuous functions agreeing a.e. on an interval agree everywhere). In particular, $F \in \mathcal{C}_{\mathcal{M}}\big((0,\delta)\big)$. Define $G(x) := F(x) - x/2$, which lies in $\mathcal{C}_{\mathcal{M}}$ by (H1).

Computing F. Integration by parts yields $\int_0^x y(-\log y)\,dy = x^2\big(\frac{1}{4}-\frac{\log x}{2}\big)$, so

$$F(x) \;=\; \frac{x^2(1/4-\log x/2)}{x(1-\log x)} \;=\; \frac{x(1-2\log x)}{4(1-\log x)}.$$

Non-constructibility. Since $\mathcal{C}_{\mathcal{M}}\big((0,1)\big)$ is an $\mathbb{R}$-algebra containing all subanalytic functions,

$$G(x) \;:=\; F(x) - \tfrac{x}{2} \;=\; \frac{-x}{4(1-\log x)}$$

would also be constructible. Setting $L = -\log x > 0$ for $x \in (0,1)$:

$$G(x) \;=\; \frac{-x}{4(1+L)} \;\sim\; -\frac{x}{4L} \;=\; -\frac{x}{4}\cdot(-\log x)^{-1} \quad \text{as } x\to 0^+.$$

By cell preparation (Recall 2.4), on some interval $(0,\delta)$, every constructible function takes the form $\sum_{i=1}^N u_i(x)\cdot x^{\alpha_i}\cdot(-\log x)^{\ell_i}$ with $\alpha_i \in \mathbb{Q}$, $\ell_i \in \mathbb{N}_0$, and u_i subanalytic units (bounded, nonvanishing). Since $G(x)/x = -1/(4(1-\log x)) \to 0$ as $x \to 0^+$:

- Terms with $\alpha_i < 1$ would make $G(x)/x$ diverge as $x\to 0^+$, contradicting $G(x)/x \to 0$.
- Terms with $\alpha_i > 1$ contribute $o(1)$ to $G(x)/x$.
- Terms with $\alpha_i = 1$ contribute $u_i(0)\cdot(-\log x)^{\ell_i}$ to $G(x)/x$. Since $\ell_i \geq 0$, each such term either diverges ($\ell_i > 0$) or approaches a nonzero constant ($\ell_i = 0$).

More precisely, grouping by leading $(-\log x)$-exponent, the asymptotic behaviour of any nonzero constructible function on $(0,\delta)$ is $Cx^{\beta}(-\log x)^{\ell}$ with $C \neq 0$, $\beta \in \mathbb{Q}$, and $\ell \in \mathbb{N}_0$; matching $G(x)/x \sim -\frac{1}{4}(-\log x)^{-1}$ would require $\ell = -1$, which cell preparation forbids. Thus G is not constructible, so F is not constructible, contradicting stabilisation. □

B Proof of Lemma 3.21

Let $\mu = \sum_i (h_i)_*(g_i \cdot \mathrm{Leb}|_{U_i})$ be a presentation with $U_i \subseteq \mathbb{R}^{m_i}$, $g_i \geq 0$, and $h_i : U_i \to \mathbb{R}^n$ definable. Each summand $\nu_i = (h_i)_*(g_i \cdot \mathrm{Leb}|_{U_i})$ is a positive measure. Since μ is absolutely continuous and $\mu = \sum_i \nu_i$ with $\nu_i \geq 0$, each ν_i must be absolutely continuous (positive singular parts cannot cancel). We show each ν_i has a constructible density, by cell decomposition of U_i into cells where h_i has constant rank.

Case 1: $\operatorname{rank} h_i < n$ **on a cell** C. The pushforward $(h_i|_C)_*(g_i \cdot \mathrm{Leb}|_C)$ is supported on $h_i(C)$, which has dimension $< n$, hence is singular with respect to Lebesgue measure on $\mathbb{R}^n$. Since ν_i is absolutely continuous, this component must be zero. It contributes nothing to the density.

Case 2: $\operatorname{rank} h_i = n$**,** $\dim U_i = n$**.** By further cell decomposition, $h_i|_C$ is injective on each sub-cell C_k. The change-of-variables formula gives pushforward density

$$f_{i,k}(z) = g_i\big((h_i|_{C_k})^{-1}(z)\big) \cdot \big|\det J_{(h_i|_{C_k})^{-1}}(z)\big|$$

at $z \in h_i(C_k)$. The inverse $(h_i|_{C_k})^{-1}$ is definable (o-minimal inverse function theorem), so $g_i \circ (h_i|_{C_k})^{-1} \in \mathcal{C}_{\mathcal{M}}$ by (H3). The Jacobian determinant of a definable map is definable, hence in $\mathcal{C}_{\mathcal{M}}$ by (H1). The product is in $\mathcal{C}_{\mathcal{M}}$ by (H1).

Case 3: $\operatorname{rank} h_i = n$**,** $\dim U_i = m_i > n$**.** After further cell decomposition by all $\binom{m_i}{n}$ minors of the Jacobian Jh_i, each resulting cell either has rank $< n$ (Case 1) or has a fixed nonvanishing $n \times n$ minor. On a cell of the second type, choose the coordinate subset I with $|I| = n$ corresponding to the nonvanishing minor, write $u = (u_I, u_{I^c})$, and define $\Phi(u) = (h_i(u), u_{I^c})$, a definable map $U_i \to \mathbb{R}^n \times \mathbb{R}^{m_i - n}$. Since $\det(\partial h_i / \partial u_I) \neq 0$ on this cell, Φ is a local C^1-diffeomorphism by the o-minimal inverse function theorem [Dri98, Ch. 7]; after further decomposition, assume $\Phi|_{C_k}$ is injective. By change of variables and Fubini:

$$f_{i,k}(z) = \int g_i(\Phi^{-1}(z,v)) \cdot |\det J\Phi^{-1}(z,v)|\, dv.$$

The integrand is in $\mathcal{C}_{\mathcal{M}}$: $g_i \circ \Phi^{-1} \in \mathcal{C}_{\mathcal{M}}$ by (H3), and $|\det J\Phi^{-1}|$ is definable, hence in $\mathcal{C}_{\mathcal{M}}$ by (H1). The integral over definable fibres gives $f_{i,k} \in \mathcal{C}_{\mathcal{M}}$ by (H4).

The total density $d\mu / d\mathrm{Leb} = \sum_{i,k} f_{i,k}$ is a finite sum of elements of $\mathcal{C}_{\mathcal{M}}$, hence in $\mathcal{C}_{\mathcal{M}}$. □

C O-minimal structures: a brief primer

This appendix collects the key definitions and examples for readers unfamiliar with o-minimal structures. We refer to van den Dries [Dri98] for a comprehensive treatment.

Definition. An *o-minimal structure* on $(\mathbb{R}, <)$ is an expansion $\mathcal{M}$ of the real ordered field $(\mathbb{R}, <, +, \cdot)$ by additional functions and relations, such that the *definable subsets of* $\mathbb{R}$ (i.e., those defined by first-order formulas in the language of $\mathcal{M}$) are exactly the finite unions of points and open intervals. Definable subsets of $\mathbb{R}^n$, and definable maps between them, are then constrained to have finite combinatorial complexity: no fractal boundaries, no space-filling curves, no pathological measure-theoretic behaviour.

Key examples.

- $(\mathbb{R}, <, +, \cdot)$: the real ordered field. Definable sets are the *semialgebraic sets* (defined by polynomial equations and inequalities). O-minimality follows from Tarski's quantifier elimination for real closed fields.
- $\mathbb{R}_{\text{an}}$: the expansion by all *restricted analytic functions* $f : [-1,1]^n \to \mathbb{R}$. Definable sets are the *globally subanalytic sets* [DD88]. This is the primary structure in this paper.
- $\mathbb{R}_{\text{an,exp}} = (\mathbb{R}_{\text{an}}, \exp)$: the further expansion by the real exponential function (equivalently, by the unrestricted logarithm). O-minimality is due to van den Dries–Miller [DM94].

Cells and cell decomposition. A *cell* in $\mathbb{R}^n$ is defined inductively. In $\mathbb{R}^1$: a point $\{a\}$ or an open interval (a,b) (allowing $a = -\infty$ or $b = +\infty$). In $\mathbb{R}^{n+1}$: either the *graph* $\{(\bar{x}, y) : \bar{x} \in C,\ y = f(\bar{x})\}$ of a continuous definable function f on a cell $C \subseteq \mathbb{R}^n$, or the *band* $\{(\bar{x}, y) : \bar{x} \in C,\ f(\bar{x}) < y < g(\bar{x})\}$ between two such functions with $f < g$ on C. The *cell decomposition theorem* [Dri98, Ch. 3] states that every definable set $X \subseteq \mathbb{R}^n$ admits a finite partition into cells, and every definable function on X is continuous on each cell of some such partition.

Closure under projections. Definable sets are closed under coordinate projections [Dri98, Ch. 3]: if $S \subseteq \mathbb{R}^{m+n}$ is definable, then $\pi_{\mathbb{R}^m}(S) \subseteq \mathbb{R}^m$ is definable. This is used throughout this paper — it ensures that relational composition preserves definability (§4) and that latent-variable constructions produce definable domains (§3).